

SUPER DUALITY AND CRYSTAL BASES FOR QUANTUM ORTHOSYMPLECTIC SUPERALGEBRAS

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ABSTRACT. We introduce a semisimple tensor category $\mathcal{O}_q^{int}(m|n)$ of modules over an quantum orthosymplectic superalgebra. It is a natural counterpart of the category of finitely dominated integrable modules over the quantum classical (super) algebra of type B_{m+n} , C_{m+n} , D_{m+n} or $B(0, m+n)$ from a viewpoint of super duality. We show that a highest weight module in $\mathcal{O}_q^{int}(m|n)$ has a unique crystal base when it corresponds to a highest weight module of type B_{m+n} , C_{m+n} or $B(0, m+n)$ under super duality. An explicit description of the crystal graph is given in terms of a new combinatorial object called orthosymplectic tableaux.

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1. INTRODUCTION

1.1. The Kashiwara's crystal base [23] is a certain nice basis at $q = 0$ of a module M over the quantized enveloping algebra associated to a symmetrizable Kac-Moody algebra, which still contains rich combinatorial information on M , and it has been one of the most important and successful tools in representation theory of the quantum groups.

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For a contragredient Lie superalgebra \mathfrak{g} [18], a theoretical background of crystal bases for the quantum superalgebra $U_q(\mathfrak{g})$ is given by Benkart, Kang and Kashiwara in [2]. The existence of a crystal base is shown when \mathfrak{g} is a general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ [2] and a queer Lie superalgebra \mathfrak{q}_n [14, 15] for a special class of finite dimensional modules appearing in a tensor power of the natural representation V of $U_q(\mathfrak{g})$, often called polynomial representations. We should remark that the crystal base theory for these two Lie superalgebras is not parallel to that of a symmetrizable Kac-Moody algebra due to the same substantial difficulties encountered when we consider the representations of classical Lie superalgebras compared to those of Lie algebras. For example, a finite dimensional representation of \mathfrak{g} is not semisimple in general. Indeed, the above results for $\mathfrak{g} = \mathfrak{gl}_{m|n}$ and \mathfrak{q}_n are based on the semisimplicity of $V^{\otimes r}$, and closely related with the Schur-Weyl-Sergeev dualities [31]. Also, there is a work on crystal bases of a family of infinite dimensional representations of $D(2|1, \alpha)$ [34].

There is another important class in classical Lie superalgebras called orthosymplectic Lie superalgebras. However, there is little known about the existence of its crystal bases except for $\mathfrak{osp}_{1|2r}$, which is a Kac-Moody superalgebra of type $B(0, r)$, and where we can apply the crystal base theory developed in [17].

In this paper, we construct for the first time crystal bases of a large family of semisimple modules over a quantum orthosymplectic Lie superalgebra. We also prove the uniqueness of these crystal bases and give a combinatorial model for the associated crystals.

1.2. Let us explain our results in more details. Our first step is to find a nice semisimple category of modules over a quantum orthosymplectic superalgebra. Since a tensor power of the natural representation of an orthosymplectic Lie superalgebra is not semisimple in general, we take a completely different approach inspired by a recent work of Cheng, Lam and Wang on super duality [7].

Super duality is an equivalence between a parabolic BGG category $\mathcal{O}(m + \infty)$ of modules over the classical Lie algebras $\mathfrak{g}_{m+\infty}$ and a category $\mathcal{O}(m|\infty)$ of modules over the basic classical Lie superalgebras $\mathfrak{g}_{m|\infty}$ of infinite rank, where $\mathfrak{g} = \mathfrak{gl}, \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$. (From now on, we use \mathfrak{g} as a symbol representing the type of a Lie superalgebra.) It was originally introduced in [8, 10] as a conjecture in case of general linear Lie superalgebras and later proved by Cheng and Lam [6]. Then the duality for orthosymplectic Lie superalgebras was established by Cheng, Lam and Wang [7]. One of its most remarkable and powerful features is that super duality reveals a natural connection with the Kazhdan-Lusztig theory of Lie algebras.

We consider the semisimple subcategory $\mathcal{O}^{int}(m|\infty)$ of $\mathcal{O}(m|\infty)$ equivalent to the subcategory $\mathcal{O}^{int}(m+\infty)$ of integrable modules in $\mathcal{O}(m+\infty)$ under super duality. It is known that $\mathcal{O}^{int}(m|\infty)$ is the category of polynomial modules when $\mathfrak{g}_{m|\infty}$ is a general linear Lie superalgebra, that is, $\mathfrak{g} = \mathfrak{gl}$, [9]. Motivated by this fact, we prove that when $\mathfrak{g}_{m|\infty}$ is orthosymplectic, that is, $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$, $\mathcal{O}^{int}(m|\infty)$ is a full subcategory of $\mathcal{O}(m|\infty)$ such that the weights of each object are polynomial with respect to a suitably chosen dual basis of the Cartan subalgebra (Theorem 3.7). We have a similar result for a category $\mathcal{O}^{int}(m|n)$ of modules over $\mathfrak{g}_{m|n}$ of finite rank, where $\mathcal{O}^{int}(m|n)$ is obtained from $\mathcal{O}^{int}(m|\infty)$ by applying the truncation functor. We should note that $\mathcal{O}^{int}(m|\infty)$ is not characterized only by locally nilpotent actions of positive simple root vectors since the odd isotropic root vectors are always nilpotent on $\mathfrak{g}_{m|n}$ -modules.

Unlike the case of $\mathfrak{g} = \mathfrak{gl}$, the irreducible modules in $\mathcal{O}^{int}(m|n)$ are infinite dimensional when $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$ and $n > 0$, which were called oscillator modules and studied via Howe duality in [5]. But one may still regard $\mathcal{O}^{int}(m|n)$ as a natural counterpart of the category $\mathcal{O}^{int}(m+n)$ of finite dimensional modules over \mathfrak{g}_{m+n} of finite rank, since both of them are obtained from two equivalent categories $\mathcal{O}^{int}(m|\infty)$ and $\mathcal{O}^{int}(m+\infty)$ by truncation, respectively (see the diagram below, where \mathcal{F} is the super duality functor and tr_n is truncation functor).

$$\begin{array}{ccc}
\mathcal{O}(m+\infty) & \xrightarrow[\mathcal{F}]{\sim} & \mathcal{O}(m|\infty) \\
\cup & & \cup \\
\mathcal{O}^{int}(m+\infty) & \xrightarrow[\mathcal{F}]{\sim} & \mathcal{O}^{int}(m|\infty) \\
\downarrow \text{tr}_n & & \downarrow \text{tr}_n \\
\mathcal{O}^{int}(m+n) & & \mathcal{O}^{int}(m|n)
\end{array}$$

Then we consider q -deformations of $\mathfrak{g}_{m|n}$ -modules in $\mathcal{O}^{int}(m|n)$ for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$. More precisely, based on our characterization of $\mathcal{O}^{int}(m|n)$, we consider a category $\mathcal{O}_q^{int}(m|n)$ of $U_q(\mathfrak{g}_{m|n})$ -modules with the same conditions on weights. By using the method of classical limit and the semisimplicity of $\mathcal{O}^{int}(m|n)$, we show the following (Theorem 4.3).

Theorem. $\mathcal{O}_q^{int}(m|n)$ is a semisimple tensor category equivalent to $\mathcal{O}^{int}(m|n)$.

An irreducible highest weight module in $\mathcal{O}_q^{int}(m|n)$ is parametrized by $\mathcal{P}(\mathfrak{g})_{m|n}$, which is a set of pairs (λ, ℓ) of a partition and a positive integer, and its highest weight is denoted by $\Lambda_{m|n}(\lambda, \ell)$. Let $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ be an irreducible highest weight $U_q(\mathfrak{g}_{m|n})$ -module with highest weight $\Lambda_{m|n}(\lambda, \ell)$.

The notion of a crystal base of a module in $\mathcal{O}_q^{int}(m|n)$ can be defined following [2] for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$. We first consider a $U_q(\mathfrak{g}_{m|n})$ -module \mathcal{V}_q with a crystal base, which is a Fock space over a q -deformed Clifford-Weyl algebra (cf.[11]). An irreducible summand of \mathcal{V}_q is a highest weight module, which corresponds to a fundamental weight $U_q(\mathfrak{g}_{m+n})$ -module under super duality, and each highest weight module in $\mathcal{O}_q^{int}(m|n)$ can be embedded into $\mathcal{V}_q^{\otimes M}$ for some $M \geq 1$.

Now, we consider the cases when $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$. To describe the connected component in the crystal of $\mathcal{V}_q^{\otimes M}$ including a highest weight vector with weight $\Lambda_{m|n}(\lambda, \ell)$, we introduce a new combinatorial object called orthosymplectic tableaux (depending on \mathfrak{g}), which is partly motivated by [32]. Let $\mathbf{T}_{m|n}(\lambda, \ell)$ denote the set of orthosymplectic tableaux of shape (λ, ℓ) . We show that the character of $\mathbf{T}_{m|n}(\lambda, \ell)$ is equal to that of $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$, and $\mathbf{T}_{m|n}(\lambda, \ell)$ is a connected crystal, where the crystal structure on $\mathbf{T}_{m|n}(\lambda, \ell)$ is naturally induced from that on $\mathcal{V}_q^{\otimes M}$. Using these facts, we prove the following, which is the main result in this paper (Theorem 8.8).

Theorem. *Each highest weight module in $\mathcal{O}_q^{int}(m|n)$ has a unique crystal base for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$. Moreover, the crystal of $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ is isomorphic to $\mathbf{T}_{m|n}(\lambda, \ell)$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$.*

We remark that an orthosymplectic tableau is defined over an arbitrary linearly ordered \mathbb{Z}_2 -graded set \mathcal{A} and its main advantage is compatibility with super duality functor \mathcal{F} and truncation functor \mathfrak{tr}_n . More precisely, the set of orthosymplectic tableaux of shape (λ, ℓ) gives both irreducible characters in $\mathcal{O}_q^{int}(m+n)$ and $\mathcal{O}_q^{int}(m|n)$ with suitable choices of \mathcal{A} . The Schur positivity of the character of orthosymplectic tableaux of shape (λ, ℓ) plays a crucial role in proving this compatibility. Also, when \mathcal{A} is a finite set with even elements, we have an explicit bijection between orthosymplectic tableaux and Kashiwara-Nakashima tableaux [26] of type B and C .

1.3. The paper is organized as follows. In Section 2, we recall the definition of orthosymplectic Lie superalgebras $\mathfrak{g}_{m|n}$ based on [9]. In Section 3, we briefly review super duality and present a simple characterization of $\mathcal{O}^{int}(m|n)$. In Section 4, we define $\mathcal{O}_q^{int}(m|n)$ and show that it is a semisimple tensor category. In Section 5, we review the notion of a crystal base for a quantum superalgebra [2] and prove the existence of a crystal base of a q -deformed Fock space \mathcal{V}_q . Then we introduce our main combinatorial object $\mathbf{T}_{m|n}(\lambda, \ell)$ in Section 6, and show that its character gives an irreducible character in $\mathcal{O}_q^{int}(m|n)$ for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$ in Section 7. Finally, in Section 8, we show that $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ has a unique crystal base for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$ and $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$, whose crystal is isomorphic to $\mathbf{T}_{m|n}(\lambda, \ell)$.

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2. ORTHOSYMPLECTIC LIE SUPERALGEBRA $\mathfrak{g}_{m|n}$

In this section, let us briefly recall some necessary background on Lie superalgebras (see [9, 18] for more details). Our exposition is based on [9] with a little modification. We assume that the base field is \mathbb{C} .

2.1. General linear Lie superalgebras. Throughout this paper, we fix a positive integer m and let

$$\begin{aligned}\widetilde{\mathbb{I}}_m &= \{ \bar{k}, -\bar{k} \mid 1 \leq k \leq m \} \cup \tfrac{1}{2}\mathbb{Z}, \\ \mathbb{I}_m &= \{ \bar{k}, -\bar{k} \mid 1 \leq k \leq m \} \cup \mathbb{Z}, \\ \bar{\mathbb{I}}_m &= \{ \bar{k}, -\bar{k} \mid 1 \leq k \leq m \} \cup (\tfrac{1}{2} + \mathbb{Z}) \cup \{0\},\end{aligned}$$

where $\widetilde{\mathbb{I}}_m$ is a linearly ordered \mathbb{Z}_2 -graded set with

$$\begin{aligned}\cdots < -\tfrac{3}{2} < -1 < -\tfrac{1}{2} < -\bar{1} < \cdots < -\bar{m} < 0 < \bar{m} < \cdots < \bar{1} < \tfrac{1}{2} < 1 < \tfrac{3}{2} < \cdots, \\ (\widetilde{\mathbb{I}}_m)_0 &\supset \{ \bar{k}, -\bar{k} \mid 1 \leq k \leq m \} \cup \mathbb{Z}^\times, \quad (\widetilde{\mathbb{I}}_m)_1 = \tfrac{1}{2} + \mathbb{Z},\end{aligned}$$

(the degree of 0 will be specified later) and the linear orderings and \mathbb{Z}_2 -gradings on the other sets are induced from those on $\widetilde{\mathbb{I}}_m$. For $a \in \widetilde{\mathbb{I}}_m$, $|a|$ denotes the degree of a . We put $\widetilde{\mathbb{I}}_m^+ = \{ a \in \widetilde{\mathbb{I}}_m^+ \mid a > 0 \}$ and $\mathbb{I}^\times = \{ a \in \mathbb{I} \mid a \neq 0 \}$ for $\mathbb{I} \subset \widetilde{\mathbb{I}}_m$.

For $\mathbb{I} \subset \widetilde{\mathbb{I}}_m$, we denote by $V_{\mathbb{I}}$ the superspace with basis $\{ v_a \mid a \in \mathbb{I} \}$, where the \mathbb{Z}_2 -grading is induced from $\widetilde{\mathbb{I}}_m$. Let $\mathfrak{gl}(V_{\mathbb{I}})$ be the general linear Lie superalgebra of linear endomorphisms on $V_{\mathbb{I}}$ vanishing on v_a 's except for finitely many a 's. We identify $\mathfrak{gl}(V_{\mathbb{I}})$ with the space of matrices $(a_{ij})_{i,j \in \mathbb{I}}$ spanned by the elementary matrices $E_{i,j}$. We assume that $V_{\mathbb{I}}$ is a subspace of $V_{\widetilde{\mathbb{I}}_m}$ and $\mathfrak{gl}(V_{\mathbb{I}}) \subset \mathfrak{gl}(V_{\widetilde{\mathbb{I}}_m})$. Let $\widehat{\mathfrak{gl}}(V_{\mathbb{I}})$ be the central extension of $\mathfrak{gl}(V_{\mathbb{I}})$ by a one-dimensional center $\mathbb{C}K$ with respect to the 2-cocycle $\alpha(A, B) = \text{Str}([J, A]B)$, where Str is the supertrace with $\text{Str}(a_{ij}) = \sum_{i \in \mathbb{I}} (-1)^{|i|} a_{ii}$ and $J = \sum_{i \leq 0} E_{i,i}$.

For $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, we put

$$\begin{aligned}\mathbb{J}_{m+n} &= \{ a \in \mathbb{I}_m \mid \bar{m} \leq a \leq n \}, \\ \mathbb{J}_{m|n} &= \{ a \in \bar{\mathbb{I}}_m \mid \bar{m} \leq a \leq n - \tfrac{1}{2} \}.\end{aligned}$$

We assume that $\mathbb{J}_{m+0} = \mathbb{J}_{m|0} = \{ \bar{m}, \dots, \bar{1} \}$.

2.2. Orthosymplectic Lie superalgebras. Suppose that the degree of $0 \in \widetilde{\mathbb{I}}_m$ is 1. Define a skew-supersymmetric bilinear form $(\cdot | \cdot)$ on $V_{\widetilde{\mathbb{I}}_m}$ by

$$(2.1) \quad \begin{aligned} (v_{\pm a} | v_{\pm b}) &= 0, & (v_a | v_{-b}) &= -(-1)^{|a||b|} (v_{-b} | v_a) = \delta_{ab}, \\ (v_0 | v_0) &= 1, & (v_0 | v_{\pm a}) &= 0, \end{aligned}$$

for $a, b \in \widetilde{\mathbb{I}}_m^+$. For $\mathbb{I} \subset \widetilde{\mathbb{I}}_m$, let $\mathfrak{spo}(V_{\mathbb{I}})$ be the subalgebra of $\mathfrak{gl}(V_{\mathbb{I}})$ preserving the skew-supersymmetric bilinear form on $V_{\mathbb{I}}$ induced from (2.1). Then we define $\mathfrak{b}_{m+\infty}^\bullet$, $\mathfrak{b}_{m|\infty}^\bullet$, $\mathfrak{c}_{m+\infty}$ and $\mathfrak{c}_{m|\infty}$ to be the central extensions of $\mathfrak{spo}(V_{\mathbb{I}})$ induced from $\widehat{\mathfrak{gl}}(V_{\mathbb{I}})$ when \mathbb{I} is \mathbb{I}_m , $\widetilde{\mathbb{I}}_m$, \mathbb{I}_m^\times and $\widetilde{\mathbb{I}}_m^\times$, respectively.

Next, suppose that the degree of $0 \in \widetilde{\mathbb{I}}_m$ is 0. Define a supersymmetric bilinear form $(\cdot | \cdot)$ on $V_{\widetilde{\mathbb{I}}_m}$ by

$$(2.2) \quad \begin{aligned} (v_{\pm a} | v_{\pm b}) &= 0, & (v_a | v_{-b}) &= (-1)^{|a||b|} (v_{-b} | v_a) = \delta_{ab}, \\ (v_0 | v_0) &= 1, & (v_0 | v_{\pm a}) &= 0, \end{aligned}$$

for $a, b \in \widetilde{\mathbb{I}}_m^+$. For $\mathbb{I} \subset \widetilde{\mathbb{I}}_m$, let $\mathfrak{osp}(V_{\mathbb{I}})$ be the subalgebra of $\mathfrak{gl}(V_{\mathbb{I}})$ preserving the supersymmetric bilinear form on $V_{\mathbb{I}}$ induced from (2.2). Then we define $\mathfrak{b}_{m+\infty}$, $\mathfrak{b}_{m|\infty}$, $\mathfrak{d}_{m+\infty}$ and $\mathfrak{d}_{m|\infty}$ to be the central extensions of $\mathfrak{osp}(V_{\mathbb{I}})$ induced from $\widehat{\mathfrak{gl}}(V_{\mathbb{I}})$ when \mathbb{I} is \mathbb{I}_m , $\widetilde{\mathbb{I}}_m$, \mathbb{I}_m^\times and $\widetilde{\mathbb{I}}_m^\times$, respectively.

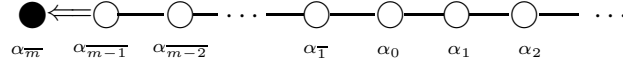
From now on, we assume that \mathfrak{g} is a symbol, which denotes one of \mathfrak{b} , \mathfrak{b}^\bullet , \mathfrak{c} and \mathfrak{d} . Let $U(\mathfrak{g}_{m+\infty})$ and $U(\mathfrak{g}_{m|\infty})$ be the enveloping superalgebras associated to $\mathfrak{g}_{m+\infty}$ and $\mathfrak{g}_{m|\infty}$, respectively.

Let $\mathfrak{h}_{m+\infty}$ (resp. $\mathfrak{h}_{m|\infty}$) be the Cartan subalgebra of $\mathfrak{g}_{m+\infty}$ (resp. $\mathfrak{g}_{m|\infty}$) spanned by K and $E_a := E_{a,a} - E_{-a,-a}$ for $a \in \mathbb{J}_{m+\infty}$ (resp. $\mathbb{J}_{m|\infty}$), and let $\mathfrak{h}_{m+\infty}^*$ (resp. $\mathfrak{h}_{m|\infty}^*$) be the restricted dual of $\mathfrak{h}_{m+\infty}$ (resp. $\mathfrak{h}_{m|\infty}$) spanned by $\Lambda_{\overline{m}}$ and δ_a for $a \in \mathbb{J}_{m+\infty}$ (resp. $\mathbb{J}_{m|\infty}$), where $\langle E_b, \delta_a \rangle = \delta_{ab}$, $\langle K, \delta_a \rangle = 0$, $\langle E_a, \Lambda_{\overline{m}} \rangle = 0$ for a, b and $\langle K, \Lambda_{\overline{m}} \rangle = r$ with $r = 1$ for $\mathfrak{g} = \mathfrak{c}$ and $r = \frac{1}{2}$ otherwise. Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing on $\mathfrak{h}_{m+\infty} \times \mathfrak{h}_{m+\infty}^*$ or $\mathfrak{h}_{m|\infty} \times \mathfrak{h}_{m|\infty}^*$.

Let $I_{m+\infty} = \{\overline{m}, \dots, \overline{1}, 0\} \cup \mathbb{Z}_{>0}$. Then the set of simple roots $\Pi_{m+\infty} = \{\alpha_i \mid i \in I_{m+\infty}\}$, the set of simple coroots $\Pi_{m+\infty}^\vee = \{\alpha_i^\vee \mid i \in I_{m+\infty}\}$ and the Dynkin diagram associated to the Cartan matrix $(\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j \in I_{m+\infty}}$ of $\mathfrak{g}_{m+\infty}$ are listed below (the simple roots are with respect to a Borel subalgebra spanned by the upper triangular matrices):

$$\bullet \mathfrak{b}_{m+\infty}^\bullet$$

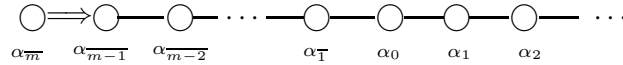
$$\alpha_i = \begin{cases} -\delta_{\overline{m}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_1, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}, \end{cases} \quad \alpha_i^\vee = \begin{cases} -2E_{\overline{m}} + 2K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} - E_1, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}. \end{cases}$$



(● denotes a non-isotropic odd simple root.)

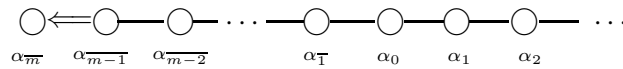
• $\mathfrak{c}_{m+\infty}$,

$$\alpha_i = \begin{cases} -2\delta_{\overline{m}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_1, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}, \end{cases} \quad \alpha_i^\vee = \begin{cases} -E_{\overline{m}} + K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} - E_1, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}. \end{cases}$$



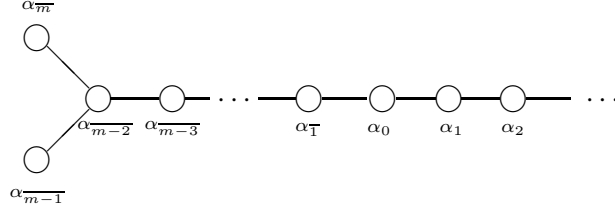
• $\mathfrak{b}_{m+\infty}$

$$\alpha_i = \begin{cases} -\delta_{\overline{m}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_1, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}, \end{cases} \quad \alpha_i^\vee = \begin{cases} -2E_{\overline{m}} + 2K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} - E_1, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}. \end{cases}$$



• $\mathfrak{d}_{m+\infty}$

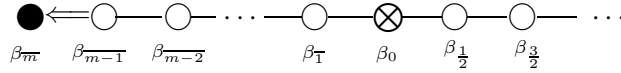
$$\alpha_i = \begin{cases} -\delta_{\overline{m}} - \delta_{\overline{m-1}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_1, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}, \end{cases} \quad \alpha_i^\vee = \begin{cases} -E_{\overline{m}} - E_{\overline{m-1}} + 2K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} - E_1, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \mathbb{Z}_{>0}. \end{cases}$$



Let $I_{m|\infty} = \{\overline{m}, \dots, \overline{1}, 0\} \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0})$. Then the set of simple roots $\Pi_{m|\infty} = \{\beta_i \mid i \in I_{m|\infty}\}$, the set of simple coroots $\Pi_{m|\infty}^\vee = \{\beta_i^\vee \mid i \in I_{m|\infty}\}$ and the Dynkin diagram associated to the Cartan matrix $(\langle \beta_i^\vee, \beta_j \rangle)_{i,j \in I_{m|\infty}}$ of $\mathfrak{g}_{m|\infty}$ are listed below (the simple roots are with respect to a Borel subalgebra spanned by the upper triangular matrices):

• $\mathfrak{b}_{m|\infty}^\bullet$

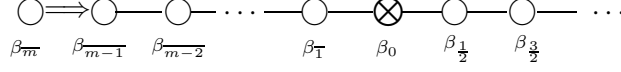
$$\beta_i = \begin{cases} -\delta_{\overline{m}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_{\frac{1}{2}}, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}, \end{cases} \quad \beta_i^\vee = \begin{cases} -2E_{\overline{m}} + 2K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} + E_{\frac{1}{2}}, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}. \end{cases}$$



(\otimes denotes an isotropic odd simple root.)

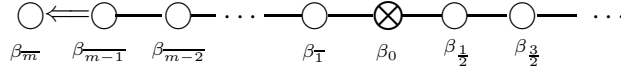
• $\mathfrak{c}_{m|\infty}$

$$\beta_i = \begin{cases} -2\delta_{\overline{m}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_{\frac{1}{2}}, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}, \end{cases} \quad \beta_i^\vee = \begin{cases} -E_{\overline{m}} + K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} + E_{\frac{1}{2}}, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}. \end{cases}$$



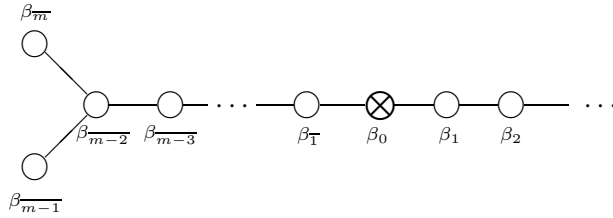
• $\mathfrak{b}_{m|\infty}$

$$\beta_i = \begin{cases} -\delta_{\overline{m}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_{\frac{1}{2}}, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}, \end{cases} \quad \beta_i^\vee = \begin{cases} -2E_{\overline{m}} + 2K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} + E_{\frac{1}{2}}, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}. \end{cases}$$



• $\mathfrak{d}_{m|\infty}$

$$\beta_i = \begin{cases} -\delta_{\overline{m}} - \delta_{\overline{m-1}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_{\frac{1}{2}}, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}, \end{cases} \quad \beta_i^\vee = \begin{cases} -E_{\overline{m}} - E_{\overline{m-1}} + 2K, & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{k} (\neq \overline{m}), \\ E_{\overline{1}} + E_{\frac{1}{2}}, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}. \end{cases}$$



Note that $\alpha_i = \beta_i$ for $i = \overline{m}, \dots, \overline{1}$.

We assume that $\mathfrak{h}_{m+\infty}^*$ and $\mathfrak{h}_{m|\infty}^*$ have symmetric bilinear forms $(\cdot | \cdot)$ given by

$$(\lambda | \delta_a) = s \langle (-1)^{|a|} E_a - K, \lambda \rangle, \quad (\Lambda_{\overline{m}} | \Lambda_{\overline{m}}) = 0,$$

for $a, b \in \mathbb{J}_{m+\infty}$ or $\mathbb{J}_{m|\infty}$ and $\lambda \in \mathfrak{h}_{m+\infty}^*$ or $\mathfrak{h}_{m|\infty}^*$. Here we assume $s = 2$ for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$, and $s = 1$ otherwise. We have $(\delta_a | \delta_b) = s(-1)^{|a|} \delta_{ab}$ for a, b , and hence

$(\alpha_i|\alpha_i), (\beta_j|\beta_j) \in 2\mathbb{Z}$ for $i \in I_{m+\infty}$ and $j \in I_{m|\infty}$. Let

$$(2.3) \quad s_i = \begin{cases} 1 & \text{if } i = \overline{m} \text{ and } \mathfrak{g} = \mathfrak{b}^\bullet, \mathfrak{b}, \mathfrak{d}, \\ 2 & \text{if } i = \overline{m} \text{ and } \mathfrak{g} = \mathfrak{c}, \\ 2 & \text{if } i \in \{\overline{m-1}, \dots, \overline{1}, 0\} \cup \mathbb{Z}_{>0} \text{ and } \mathfrak{g} = \mathfrak{b}^\bullet, \mathfrak{b}, \\ 1 & \text{if } i \in \{\overline{m-1}, \dots, \overline{1}, 0\} \cup \mathbb{Z}_{>0} \text{ and } \mathfrak{g} = \mathfrak{c}, \mathfrak{d}, \\ -2 & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0} \text{ and } \mathfrak{g} = \mathfrak{b}^\bullet, \mathfrak{b}, \\ -1 & \text{if } i \in \frac{1}{2} + \mathbb{Z}_{\geq 0} \text{ and } \mathfrak{g} = \mathfrak{c}, \mathfrak{d}. \end{cases}$$

Then $s_i \langle \alpha_i^\vee, \lambda \rangle = (\alpha_i|\lambda)$ for $i \in I_{m+\infty}$, $\lambda \in \mathfrak{h}_{m+\infty}^*$, and $s_j \langle \beta_j^\vee, \mu \rangle = (\beta_j|\mu)$ for $j \in I_{m|\infty}$, $\mu \in \mathfrak{h}_{m|\infty}^*$.

For $n \geq 0$, we put $I_{m+n} = \{i \in I_{m+\infty} \mid (\alpha_i|\delta_a) \neq 0 \text{ for some } a \in \mathbb{J}_{m+n}\}$ and $I_{m|n} = \{i \in I_{m|\infty} \mid (\beta_i|\delta_a) \neq 0 \text{ for some } a \in \mathbb{J}_{m|n}\}$. Let \mathfrak{g}_{m+n} (resp. $\mathfrak{g}_{m|n}$) be the subalgebra of $\mathfrak{g}_{m+\infty}$ (resp. $\mathfrak{g}_{m|\infty}$) generated by the root vectors $E_{\pm\gamma}$ for $\gamma \in \Pi_{m+n} := \{\alpha_i \mid i \in I_{m+n}\}$ (resp. $\Pi_{m|n} := \{\beta_i \mid i \in I_{m|n}\}$) and K . The Cartan subalgebra \mathfrak{h}_{m+n} (resp. $\mathfrak{h}_{m|n}$) of \mathfrak{g}_{m+n} (resp. $\mathfrak{g}_{m|n}$) is spanned by K and E_a for $a \in \mathbb{J}_{m+n}$ (resp. $\mathbb{J}_{m|n}$).

3. SUPER DUALITY AND A SEMISIMPLE TENSOR CATEGORY OF $\mathfrak{g}_{m|n}$ -MODULES

Throughout the paper, a module M over a superalgebra U is understood to be a supermodule, that is, $M = M_0 \oplus M_1$ with $U_i M_j \subset M_{i+j}$ for $i, j \in \mathbb{Z}_2$. If U has a comultiplication Δ , then we have a U -module structure on $M \otimes N$ via Δ for U -modules M and N , where we have a superalgebra structure on $U \otimes U$ with multiplication $(u_1 \otimes u_2)(v_1 \otimes v_2) = (-1)^{|u_2||v_1|}(u_1 v_1) \otimes (u_2 v_2)$ ($|u|$ denotes the degree of a homogeneous element $u \in U$).

3.1. Super duality. Let us briefly recall the *super duality* for orthosymplectic Lie superalgebras [7]. Let \mathcal{P} denote the set of partitions. For $\lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P}$, let $\lambda' = (\lambda'_i)_{i \geq 1}$ be the conjugate of λ .

Let $\mathfrak{l}_{m+\infty}$ be the standard Levi subalgebra of $\mathfrak{g}_{m+\infty}$ corresponding to $\{\alpha_i \mid i \in J_{m+\infty}\}$ for some $J_{m+\infty}$ with $\mathbb{Z}_{>0} \subset J_{m+\infty} \subset I_{m+\infty} \setminus \{0\}$. Let

$$P_{m+\infty}^+ = \left\{ \Lambda = c\Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m+\infty}} \lambda_a \delta_a \left| \begin{array}{l} (1) \ c \in \mathbb{C} \text{ and } \lambda^+ := (\lambda_1, \lambda_2, \dots) \in \mathcal{P}, \\ (2) \ \langle \alpha_i^\vee, \Lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for } i \in J_{m+\infty} \end{array} \right. \right\}$$

be the set of $\mathfrak{l}_{m+\infty}$ -dominant integral weights in $\mathfrak{h}_{m+\infty}^*$. For $\Lambda \in P_{m+\infty}^+$, let $L(\mathfrak{l}_{m+\infty}, \Lambda)$ be the irreducible $\mathfrak{l}_{m+\infty}$ -module with highest weight Λ , and $L(\mathfrak{g}_{m+\infty}, \Lambda)$

the irreducible quotient of $K(\mathfrak{g}_{m+\infty}, \Lambda) := \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}_{m+\infty}} L(\mathfrak{l}_{m+\infty}, \Lambda)$, where \mathfrak{p} is the subalgebra spanned by K , upper triangular matrices and $\mathfrak{l}_{m+\infty}$, and $L(\mathfrak{l}_{m+\infty}, \Lambda)$ is extended to a \mathfrak{p} -module in a trivial way.

Let $\mathcal{O}(m+\infty)$ be the category of $\mathfrak{g}_{m+\infty}$ -modules M satisfying

- (1) $M = \bigoplus_{\gamma \in \mathfrak{h}_{m+\infty}^*} M_\gamma$ and $\dim M_\gamma < \infty$ for $\gamma \in \mathfrak{h}_{m+\infty}^*$,
- (2) $\text{wt}(M) \subset \bigcup_{i=1}^r \left(\Lambda_i - \sum_{\Pi_{m+\infty}} \mathbb{Z}_{\geq 0} \alpha \right)$ for some $r \geq 1$ and $\Lambda_i \in P_{m+\infty}^+$,
- (3) M decomposes into a direct sum of $L(\mathfrak{l}_{m+\infty}, \Lambda)$'s for $\Lambda \in P_{m+\infty}^+$.

Here $M_\gamma = \{ m \mid h \cdot m = \langle h, \gamma \rangle m \text{ } (h \in \mathfrak{h}_{m+\infty}) \}$ and $\text{wt}(M) = \{ \gamma \in \mathfrak{h}_{m+\infty}^* \mid M_\gamma \neq 0 \}$ called the set of weights of M . Note that (3) can be replaced by the condition that $E_{-\alpha_i}$ is locally nilpotent on M for $i \in J_{m+\infty}$, that is, M is an integrable $\mathfrak{l}_{m+\infty}$ -module, where $E_{-\alpha_i}$ denotes as usual a non-zero root vector associated to $-\alpha_i$ (see [19, Section 2.5] for the case of $\mathfrak{osp}(1|2m)$ -modules).

Next, let $\mathfrak{l}_{m|\infty}$ be the standard Levi subalgebra of $\mathfrak{g}_{m|\infty}$ corresponding to $\{ \beta_i \mid i \in J_{m|\infty} \}$, where $J_{m|\infty} = (J_{m+\infty} \cap \{ \overline{m}, \dots, \overline{1} \}) \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0})$. Let

$$P_{m|\infty}^+ = \left\{ \Lambda^\natural \mid \Lambda \in P_{m+\infty}^+ \right\},$$

where $\Lambda^\natural = c\Lambda_{\overline{m}} + \sum_{a=\overline{m}}^{\overline{1}} \lambda_a \delta_a + \sum_{b \in \frac{1}{2} + \mathbb{Z}_{\geq 0}} (\lambda^+)'_{b+\frac{1}{2}} \delta_b$ for $\Lambda = c\Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m+\infty}} \lambda_a \delta_a$. Then we define $L(\mathfrak{l}_{m|\infty}, \Lambda^\natural)$, $K(\mathfrak{g}_{m|\infty}, \Lambda^\natural)$, and $L(\mathfrak{g}_{m|\infty}, \Lambda^\natural)$ for $\Lambda \in P_{m+\infty}^+$ in the same way as in $\mathcal{O}(m+\infty)$.

Let $\mathcal{O}(m|\infty)$ be the category of $\mathfrak{g}_{m|\infty}$ -modules M satisfying

- (1) $M = \bigoplus_{\gamma \in \mathfrak{h}_{m|\infty}^*} M_\gamma$ and $\dim M_\gamma < \infty$ for $\gamma \in \mathfrak{h}_{m|\infty}^*$,
- (2) $\text{wt}(M) \subset \bigcup_{i=1}^r \left(\Lambda_i - \sum_{\Pi_{m|\infty}} \mathbb{Z}_{\geq 0} \beta \right)$ for some $r \geq 1$ and $\Lambda_i \in P_{m|\infty}^+$,
- (3) M decomposes into a direct sum of $L(\mathfrak{l}_{m|\infty}, \Lambda)$'s for $\Lambda \in P_{m|\infty}^+$.

Remark 3.1. For $\Lambda = c\Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|\infty}} \lambda_a \delta_a \in \text{wt}(M)$, we assume that the parity of Λ is $\sum_{a \geq \frac{1}{2}} \lambda_a \pmod{2}$ when $\mathfrak{g} \neq \mathfrak{b}^\bullet$, and $\sum_{a=\overline{m}}^{\overline{1}} \lambda_a \pmod{2}$ when $\mathfrak{g} = \mathfrak{b}^\bullet$, which we denote by $|\Lambda|$ (cf. [7, Section 5.2]). In particular, we have $|\beta_i| = 0$ (resp. 1) if and only if β_i is even (resp. odd) for $i \in I_{m|\infty}$. We assume that the \mathbb{Z}_2 -grading on $M \in \mathcal{O}(m|\infty)$ is induced from the parity of its weights.

Note that $\{ L(\mathfrak{g}_{m+\infty}, \Lambda) \mid \Lambda \in P_{m+\infty}^+ \}$ and $\{ L(\mathfrak{g}_{m|\infty}, \Lambda^\natural) \mid \Lambda \in P_{m+\infty}^+ \}$ form complete lists of irreducibles (up to isomorphism) in $\mathcal{O}(m+\infty)$ and $\mathcal{O}(m|\infty)$, respectively. By [7, Theorems 4.6 and 5.4], we have the following, which is called *super duality*.

Theorem 3.2. *There exists an equivalence of categories $\mathcal{F} : \mathcal{O}(m+\infty) \longrightarrow \mathcal{O}(m|\infty)$ such that $\mathcal{F}(L(\mathfrak{g}_{m+\infty}, \Lambda)) \cong L(\mathfrak{g}_{m|\infty}, \Lambda^\natural)$ for $\Lambda \in P_{m+\infty}^+$*

Let us give a brief description of \mathcal{F} for the readers' convenience. The super duality [7] is indeed established by considering an intermediate category between $\mathcal{O}(m+\infty)$ and $\mathcal{O}(m|\infty)$, which plays a crucial role.

Let $\tilde{\mathfrak{b}}^\bullet$, $\tilde{\mathfrak{c}}$ and $\tilde{\mathfrak{b}}$, $\tilde{\mathfrak{d}}$ be the central extensions of $\mathfrak{spo}(V_{\mathbb{I}})$ and $\mathfrak{osp}(V_{\mathbb{I}})$ with respect to the 2-cocycle α for $\mathbb{I} = \tilde{\mathbb{I}}_m, \tilde{\mathbb{I}}_m^\times$, respectively. Let $\tilde{\mathfrak{h}}$ be the Cartan subalgebra of $\tilde{\mathfrak{g}}$ with basis $\{K, E_a \ (a \in \tilde{\mathbb{I}}_m^+)\}$, and $\tilde{\mathfrak{h}}^*$ the restricted dual spanned by $\{\Lambda_{\overline{m}}, \delta_a \ (a \in \tilde{\mathbb{I}}_m^+)\}$. The set of simple roots of $\tilde{\mathfrak{g}}$ is $\tilde{\Pi} = \{\gamma_i \mid i \in \tilde{I}\}$, where $\tilde{I} = \{\overline{m}, \dots, \overline{1}, 0\} \cup \frac{1}{2}\mathbb{Z}_{>0}$, and $\gamma_i = \beta_i \ (i = \overline{m}, \dots, \overline{1}, 0)$, $\gamma_j = \delta_j - \delta_{j+\frac{1}{2}} \ (r \in \frac{1}{2}\mathbb{Z}_{>0})$. By definition, we have $\mathfrak{g}_{m+\infty}, \mathfrak{g}_{m|\infty} \subset \tilde{\mathfrak{g}}$.

Let $\tilde{\mathfrak{l}}$ be the standard Levi subalgebra of $\tilde{\mathfrak{g}}$ corresponding to $\{\gamma_i \mid i \in \tilde{J}\}$ with $\tilde{J} = (J_{m+\infty} \cap \{\overline{m}, \dots, \overline{1}\}) \cup \frac{1}{2}\mathbb{Z}_{>0}$, and let $\tilde{P}^+ = \{\Lambda^\theta \mid \Lambda \in P_{m+\infty}^+\}$, where $\Lambda^\theta = c\Lambda_{\overline{m}} + \sum_{a=\overline{m}}^{\overline{1}} \lambda_a \delta_a + \sum_{b \in \frac{1}{2}\mathbb{Z}_{>0}} \theta(\lambda^+)_b \delta_b$, with $\theta(\lambda^+)_{i-\frac{1}{2}} = \max\{\lambda'_i - i + 1, 0\}$ and $\theta(\lambda^+)_i = \max\{\lambda_i - i, 0\} \ (i \in \mathbb{Z}_{>0})$ for $\Lambda = c\Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m+\infty}} \lambda_a \delta_a$. Define $L(\tilde{\mathfrak{l}}, \Lambda^\theta)$, $K(\tilde{\mathfrak{g}}, \Lambda^\theta)$, and $L(\tilde{\mathfrak{g}}, \Lambda^\theta)$ for $\Lambda \in P_{m+\infty}^+$ as in $\mathcal{O}(m+\infty)$ and $\mathcal{O}(m|\infty)$. Let $\tilde{\mathcal{O}}$ be the category of $\tilde{\mathfrak{g}}$ -modules M such that (1) M is $\tilde{\mathfrak{h}}$ -semisimple with finite dimensional weight spaces, (2) $\text{wt}(M)$ is dominated by a finite number of weights in \tilde{P}^+ , (3) M is a direct sum of $L(\tilde{\mathfrak{l}}, \Lambda^\theta)$'s (see [7, Section 3.2] for more detail).

Now, for $M = \bigoplus_{\gamma \in \tilde{\mathfrak{h}}^*} M_\gamma \in \tilde{\mathcal{O}}$, define

$$T(M) = \bigoplus_{\gamma \in \mathfrak{h}_{m+\infty}^*} M_\gamma, \quad \overline{T}(M) = \bigoplus_{\gamma \in \mathfrak{h}_{m|\infty}^*} M_\gamma.$$

Then we have equivalences of categories $T : \tilde{\mathcal{O}} \longrightarrow \mathcal{O}(m+\infty)$ and $\overline{T} : \tilde{\mathcal{O}} \longrightarrow \mathcal{O}(m|\infty)$ [7, Theorem 5.4] such that

$$(3.1) \quad \begin{cases} T(L(\tilde{\mathfrak{l}}, \Lambda^\theta)) = L(\mathfrak{l}_{m+\infty}, \Lambda), & \begin{cases} \overline{T}(L(\tilde{\mathfrak{l}}, \Lambda^\theta)) = L(\mathfrak{l}_{m|\infty}, \Lambda^\natural), \\ \overline{T}(X(\tilde{\mathfrak{g}}, \Lambda^\theta)) = X(\mathfrak{g}_{m|\infty}, \Lambda^\natural), \end{cases} \\ T(X(\tilde{\mathfrak{g}}, \Lambda^\theta)) = X(\mathfrak{g}_{m+\infty}, \Lambda), \end{cases}$$

for $X = K, L$. The functor \mathcal{F} in Theorem 3.2 is understood to be $\overline{T} \circ S$, where $S : \mathcal{O}(m+\infty) \longrightarrow \tilde{\mathcal{O}}$ is a functor such that $S \circ T$ (resp. $T \circ S$) is naturally isomorphic to the identity functor $1_{\tilde{\mathcal{O}}}$ (resp. $1_{\mathcal{O}(m+\infty)}$).

$$(3.2) \quad \begin{array}{ccc} & \tilde{\mathcal{O}} & \\ T \swarrow & & \searrow \overline{T} \\ \mathcal{O}(m+\infty) & \xrightarrow{\mathcal{F}} & \mathcal{O}(m|\infty) \end{array}$$

Remark 3.3.

(1) Our exposition is a special case of the results in [7], since we assume here that $m > 0$.

(2) In [7], the authors use a central extension, say $\widehat{\mathfrak{gl}}'(V_{\mathbb{I}_m})$, of $\mathfrak{gl}(V_{\mathbb{I}_m})$ by a one-dimensional center $\mathbb{C}K$ with respect to the 2-cocycle $\alpha'(X, Y) = \text{Str}([J', X]Y)$, where $J' = E_{0,0} + \sum_{r \leq -1} E_{r,r}$ in order to describe in a unified way the truncation into modules over the orthosymplectic Lie superalgebras of finite rank. On the other hand, our central extension $\widehat{\mathfrak{gl}}(V_{\mathbb{I}_m})$ is given by using $J = \sum_{r \leq 0} E_{r,r}$ to describe the fundamental weight $\Lambda_{\overline{m}}$ for $\mathfrak{g}_{m+\infty}$ and $\mathfrak{g}_{m|\infty}$. But, there is an isomorphism $\psi : \widehat{\mathfrak{gl}}'(V_{\mathbb{I}_m}) \rightarrow \widehat{\mathfrak{gl}}(V_{\mathbb{I}_m})$ given by $\psi(X) = X + \text{Str}(J''X)K$ for $X \in \mathfrak{gl}(V_{\mathbb{I}_m})$ and $\psi(K) = K$, where $J'' = J - J'$ (cf. [7, Section 2.4]). So by using ψ one can translate the results in [7] in terms of our setting without difficulty.

3.2. The category $\mathcal{O}^{int}(m|\infty)$. For $\lambda \in \mathcal{P}$ and $c \in \mathbb{C}$, put

$$\begin{aligned}\Lambda_{m+\infty}(\lambda, c) &= c\Lambda_{\overline{m}} + \lambda_1\delta_{\overline{m}} + \cdots + \lambda_m\delta_{\overline{1}} + \lambda_{m+1}\delta_1 + \lambda_{m+2}\delta_2 + \cdots, \\ \Lambda_{m|\infty}(\lambda, c) &= \Lambda_{m+\infty}(\lambda, c)^{\natural}.\end{aligned}$$

Let $\mathcal{P}(\mathfrak{g})$ be given by

$$\begin{aligned}\mathcal{P}(\mathfrak{b}^\bullet) &= \{(\lambda, \ell) \in \mathcal{P} \times \mathbb{Z}_{>0} \mid \ell - 2\lambda_1 \in 2\mathbb{Z}_{\geq 0}\}, \\ \mathcal{P}(\mathfrak{c}) &= \{(\lambda, \ell) \in \mathcal{P} \times \mathbb{Z}_{>0} \mid \ell - \lambda_1 \in \mathbb{Z}_{\geq 0}\}, \\ \mathcal{P}(\mathfrak{b}) &= \{(\lambda, \ell) \in \mathcal{P} \times \mathbb{Z}_{>0} \mid \ell - 2\lambda_1 \in \mathbb{Z}_{\geq 0}\}, \\ \mathcal{P}(\mathfrak{d}) &= \{(\lambda, \ell) \in \mathcal{P} \times \mathbb{Z}_{>0} \mid \ell - \lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0}\}.\end{aligned}$$

Let $\mathcal{O}^{int}(m+\infty)$ be a full subcategory of $\mathfrak{g}_{m+\infty}$ -modules M in $\mathcal{O}(m+\infty)$ such that $E_{-\alpha}$ is locally nilpotent on M for $\alpha \in \Pi_{m+\infty}$. It is well-known that $\mathcal{O}^{int}(m+\infty)$ is a semisimple tensor category, whose irreducible objects are $L(\mathfrak{g}_{m+\infty}, \Lambda_{m+\infty}(\lambda, \ell))$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$ (see [19, Section 2.5] when $\mathfrak{g} = \mathfrak{b}^\bullet$).

Now, we want to characterize the $\mathfrak{g}_{m|\infty}$ -modules which correspond to integrable $\mathfrak{g}_{m+\infty}$ -modules in $\mathcal{O}^{int}(m+\infty)$ under the super duality functor \mathcal{F} . Since $E_{-\beta_0}^2 = 0$ and hence $E_{-\beta_0}$ is always locally nilpotent on $M \in \mathcal{O}(m|\infty)$, we do not necessarily obtain such $\mathfrak{g}_{m|\infty}$ -modules by the condition that $E_{-\beta}$ is locally nilpotent on M for all $\beta \in \Pi_{m|\infty}$.

Definition 3.4. Define $\mathcal{O}^{int}(m|\infty)$ to be the category of $\mathfrak{g}_{m|\infty}$ -modules M satisfying the following conditions:

- (1') $M = \bigoplus_{\gamma \in \mathfrak{h}_{m|\infty}^*} M_\gamma$ and $\dim M_\gamma < \infty$ for $\gamma \in \mathfrak{h}_{m|\infty}^*$
- (2') $\text{wt}(M) \subset \bigcup_{i=1}^r \left(\ell_i \Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|\infty}} \mathbb{Z}_{\geq 0} \delta_a \right)$ for some $r \geq 1$ and $\ell_i \in \mathbb{Z}_{\geq 0}$,
- (3') $E_{-\beta_{\overline{m}}}$ is locally nilpotent on M .

Note that the condition (2') implies the condition (2) in $\mathcal{O}(m|\infty)$ and also implies that $E_{-\beta_i}$ is locally nilpotent on M for $i \neq \overline{m}, 0$, which combined with (3') guarantees that M is a direct sum of $L(\mathfrak{l}_{m|\infty}, \Lambda)$'s for $\Lambda \in P_{m|\infty}^+$. Hence $M \in \mathcal{O}(m|\infty)$. Equivalently, $\mathcal{O}^{int}(m|\infty)$ is a full subcategory of $\mathfrak{g}_{m|\infty}$ -modules M in $\mathcal{O}(m|\infty)$ with

$$(3.3) \quad \text{wt}(M) \subset \mathbb{Z}_{\geq 0} \Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|\infty}} \mathbb{Z}_{\geq 0} \delta_a.$$

The goal of this section is to show that $\mathcal{O}^{int}(m|\infty)$ is a semisimple tensor category, which is equivalent to $\mathcal{O}^{int}(m+\infty)$ under \mathcal{F} . For this, we need the following two lemmas.

Let $\mathfrak{gl}_{m+\infty}$ be the standard Levi subalgebra of $\mathfrak{g}_{m+\infty}$ corresponding to $\Pi_{m+\infty} \setminus \{\alpha_{\overline{m}}\}$. Let $\mathfrak{h}_{m+\infty}^\circ$ be the Cartan subalgebra of $\mathfrak{gl}_{m+\infty}$ with basis $\{E_a \mid a \in \mathbb{J}_{m+\infty}\}$, and $(\mathfrak{h}_{m+\infty}^\circ)^*$ the restricted dual spanned by dual basis $\{\delta_a \mid a \in \mathbb{J}_{m+\infty}\}$. Also, let $\mathfrak{gl}_{m|\infty}$ be the standard Levi subalgebra of $\mathfrak{g}_{m|\infty}$ corresponding to $\Pi_{m|\infty} \setminus \{\beta_{\overline{m}}\}$ with $\mathfrak{h}_{m|\infty}^\circ$ and $(\mathfrak{h}_{m|\infty}^\circ)^*$ defined in a similar way.

For $\lambda \in \mathcal{P}$, let $L(\mathfrak{gl}_{m+\infty}, \lambda)$ and $L(\mathfrak{gl}_{m|\infty}, \lambda^\natural)$ be the irreducible highest weight modules over $\mathfrak{gl}_{m+\infty}$ and $\mathfrak{gl}_{m|\infty}$ with highest weight $\Lambda_{m+\infty}(\lambda, 0)$ and $\Lambda_{m+\infty}(\lambda, 0)^\natural$, respectively.

Lemma 3.5. *For $M \in \mathcal{O}^{int}(m|\infty)$, M is a direct sum of $L(\mathfrak{gl}_{m|\infty}, \lambda^\natural)$'s for $\lambda \in \mathcal{P}$.*

Proof. By (3.3) and [9, Theorem 3.27] (cf. [4, Section 3.2.2]), M is a polynomial representation of $\mathfrak{gl}_{m|\infty}$ and hence completely reducible. That is, M is a direct sum of $L(\mathfrak{gl}_{m|\infty}, \lambda^\natural)$'s for $\lambda \in \mathcal{P}$. \square

Let $\mathcal{G} : \mathcal{O}(m|\infty) \rightarrow \mathcal{O}(m+\infty)$ be an equivalence of categories such that $\mathcal{G} \circ \mathcal{F}$ (resp. $\mathcal{F} \circ \mathcal{G}$) is naturally isomorphic to $\text{id}_{\mathcal{O}(m+\infty)}$ (resp. $\text{id}_{\mathcal{O}(m|\infty)}$).

Lemma 3.6. *For $M \in \mathcal{O}^{int}(m|\infty)$, $\mathcal{G}(M)$ is a direct sum of $L(\mathfrak{gl}_{m+\infty}, \lambda)$'s for $\lambda \in \mathcal{P}$.*

Proof. Let us briefly recall the super duality for general linear Lie superalgebras [6] (with respect to a maximal Levi subalgebra). Let $P_{m+\infty}^{+'}$ be the set of weights $\Lambda = \sum_{a \in \mathbb{J}_{m+\infty}} \lambda_a \delta_a \in \mathfrak{h}_{m+\infty}^*$ such that $\lambda^+ = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$ and $\langle \alpha_i^\vee, \Lambda \rangle \in \mathbb{Z}_{\geq 0}$ for $i \in I_{m+\infty} \setminus \{\overline{m}, 0\}$. Let $\mathcal{O}_{\mathfrak{gl}_{m+\infty}}$ be the category of $\mathfrak{gl}_{m+\infty}$ -modules M satisfying

- (1) $M = \bigoplus_{\gamma \in (\mathfrak{h}_{m+\infty}^\circ)^*} M_\gamma$ and $\dim M_\gamma < \infty$ for $\gamma \in (\mathfrak{h}_{m+\infty}^\circ)^*$,
- (2) $\text{wt}(M) \subset \bigcup_{i=1}^r \left(\Lambda_i - \sum_{\Pi_{m+\infty} \setminus \{\alpha_{\overline{m}}\}} \mathbb{Z}_{\geq 0} \alpha \right)$ for some $r \geq 1$ and $\Lambda_i \in P_{m+\infty}^{+'}$,
- (3) M decomposes into a direct sum of $L(\mathfrak{l}_{m+\infty}, \Lambda)$'s for $\Lambda \in P_{m+\infty}^{+'}$,

where $\text{wt}(M)$ is the set of weights of M with respect to $\mathfrak{h}_{m+\infty}^\circ$, and $\mathfrak{l}_{m+\infty}'$ is the standard Levi subalgebra corresponding to $\Pi_{m+\infty} \setminus \{\alpha_{\overline{m}}, \alpha_0\}$. Also we define the category $\mathcal{O}_{\mathfrak{gl}_{m|\infty}}$ of $\mathfrak{gl}_{m|\infty}$ -modules in a similar fashion.

Let $\widetilde{\mathfrak{gl}}$ be the standard Levi subalgebra of $\widetilde{\mathfrak{g}}$ corresponding to $\widetilde{\Pi} \setminus \{\gamma_{\overline{m}}\}$, and let $\mathcal{O}_{\widetilde{\mathfrak{gl}}}$ be a parabolic category of $\widetilde{\mathfrak{gl}}$ -modules with respect to $\widetilde{\Pi} \setminus \{\gamma_{\overline{m}}, \gamma_0\}$ defined in the same way as $\mathcal{O}_{\mathfrak{gl}_{m+\infty}}$ or $\mathcal{O}_{\mathfrak{gl}_{m|\infty}}$ (see [6] for more detail). Let $\widetilde{\mathfrak{h}}^\circ$ be the Cartan subalgebra of $\widetilde{\mathfrak{gl}}$ with basis $\{E_a \mid a \in \widetilde{\mathbb{I}}_m^+\}$ and $(\widetilde{\mathfrak{h}}^\circ)^*$ its restricted dual spanned by $\{\delta_a \mid a \in \widetilde{\mathbb{I}}_m^+\}$. By [6, Theorem 5.1] (see also [9, Theorem 6.38]), we have equivalences of categories

$$(3.4) \quad \mathcal{O}_{\mathfrak{gl}_{m+\infty}} \xleftarrow{T'} \mathcal{O}_{\widetilde{\mathfrak{gl}}} \xrightarrow{\overline{T}'} \mathcal{O}_{\mathfrak{gl}_{m|\infty}},$$

where for $M = \bigoplus_{\gamma \in (\widetilde{\mathfrak{h}}^\circ)^*} M_\gamma \in \mathcal{O}_{\widetilde{\mathfrak{gl}}}$,

$$T'(M) = \bigoplus_{\gamma \in (\mathfrak{h}_{m+\infty}^\circ)^*} M_\gamma, \quad \overline{T}'(M) = \bigoplus_{\gamma \in (\mathfrak{h}_{m|\infty}^\circ)^*} M_\gamma.$$

Now, let $\overline{M} \in \mathcal{O}(m|\infty)$ be given. Since $\overline{M} \cong \overline{T}(\widetilde{M})$ for some $\widetilde{M} \in \widetilde{\mathcal{O}}$ by (3.2), let us identify \overline{M} with $T(\widetilde{M})$ and put $M = T(\widetilde{M})$. Note that $M \cong \mathcal{G}(\overline{M})$ by Theorem 3.2. By definition of $\widetilde{\mathcal{O}}$, we have $\text{wt}(\widetilde{M}) \subset \bigcup_{i=1}^r D(\Lambda_i)$ for some $\Lambda_1, \dots, \Lambda_r \in \widetilde{P}^+$, where $D(\Lambda_i) = \Lambda_i - \sum_{\widetilde{\Pi}} \mathbb{Z}_{\geq 0} \gamma$. We may assume that $D(\Lambda_i) \cap D(\Lambda_j) = \emptyset$ for $i \neq j$.

For $\Lambda \in \widetilde{P}^+$ and $k \geq 1$, put $D(\Lambda)_k = \Lambda - k\gamma_{\overline{m}} - \sum_{\widetilde{\Pi} \setminus \{\gamma_{\overline{m}}\}} \mathbb{Z}_{\geq 0} \gamma$. Then we have $\widetilde{M} = \bigoplus_{k \geq 0} \widetilde{M}_k$, where \widetilde{M}_k is the sum of \widetilde{M}_μ over $\mu \in \text{wt}(\widetilde{M})$ such that $\mu \in \bigcup_{i=1}^r D(\Lambda_i)_k$. We can check that \widetilde{M}_k is $\widetilde{\mathfrak{gl}}$ -invariant and $\widetilde{M}_k \in \mathcal{O}_{\widetilde{\mathfrak{gl}}}$ since its weights are finitely dominated as an $\widetilde{\mathfrak{gl}}$ -module. Hence we have

$$(3.5) \quad \overline{M} = \bigoplus_{k \geq 0} \overline{M}_k, \quad M = \bigoplus_{k \geq 0} M_k,$$

where $\overline{M}_k := \overline{T}(\widetilde{M}_k) \in \mathcal{O}_{\mathfrak{gl}_{m|\infty}}$, and $M_k := \overline{T}(\widetilde{M}_k) \in \mathcal{O}_{\mathfrak{gl}_{m+\infty}}$. Also, we observe that \overline{T} and T when restricted on \widetilde{M}_k coincide with \overline{T}' and T' , respectively.

Now, suppose that $\overline{M} \in \mathcal{O}^{int}(m|\infty)$. By Lemma 3.5, \overline{M}_k is a direct sum of $L(\mathfrak{gl}_{m|\infty}, \lambda^\natural)$'s ($\lambda \in \mathcal{P}$). By (3.4), M_k is a direct sum of $L(\mathfrak{gl}_{m+\infty}, \lambda)$'s ($\lambda \in \mathcal{P}$) with the same multiplicity as \overline{M}_k for each λ . Finally, it follows from (3.5) that $M \cong \mathcal{G}(\overline{M})$ is a direct sum of $L(\mathfrak{gl}_{m+\infty}, \lambda)$'s ($\lambda \in \mathcal{P}$). \square

Theorem 3.7. $\mathcal{O}^{int}(m|\infty)$ is a semisimple tensor category, whose irreducible objects are $L(\mathfrak{gl}_{m|\infty}, \Lambda_{m|\infty}(\lambda, \ell))$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$.

Proof. Suppose that $M \in \mathcal{O}^{int}(m|\infty)$ is given. By the condition (2'), we may assume that $M \in \mathcal{O}(m|\infty)$ with $J_{m|\infty} = I_{m|\infty} \setminus \{0\}$. We first claim that $\mathcal{G}(M) \in$

$\mathcal{O}^{int}(m+\infty)$. By definition of $\mathcal{O}(m+\infty)$, $\mathcal{G}(M)$ is an integrable $\mathfrak{l}_{m+\infty}$ -module. Hence $E_{-\alpha_i}$ is locally nilpotent on $\mathcal{G}(M)$ for $i \in J_{m+\infty}$. On the other hand, by Lemma 3.6, $\mathcal{G}(M)$ is a direct sum of $L(\mathfrak{gl}_{m+\infty}, \lambda)$'s ($\lambda \in \mathcal{P}$). In particular, $E_{-\alpha_0}$ acts locally nilpotently on $\mathcal{G}(M)$. Therefore, $\mathcal{G}(M) \in \mathcal{O}^{int}(m+\infty)$. Since $\mathcal{O}^{int}(m+\infty)$ is semisimple, we have $\mathcal{G}(M) \cong \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})} L(\mathfrak{gl}_{m+\infty}, \Lambda_{m+\infty}(\lambda, \ell))^{\oplus m_{(\lambda, \ell)}}$, for some $m_{(\lambda, \ell)} \in \mathbb{Z}_{\geq 0}$. Then applying \mathcal{F} to $\mathcal{G}(M)$, we get

$$M \cong \bigoplus_{(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})} L(\mathfrak{gl}_{m|\infty}, \Lambda_{m|\infty}(\lambda, \ell))^{\oplus m_{(\lambda, \ell)}}.$$

Hence M is semisimple. Finally, given $M_1, M_2 \in \mathcal{O}^{int}(m|\infty)$, it is not difficult to check that $M_1 \otimes M_2 \in \mathcal{O}^{int}(m|\infty)$. Therefore, $\mathcal{O}^{int}(m|\infty)$ is a semisimple tensor category. \square

Corollary 3.8. *Let M be a highest weight $\mathfrak{gl}_{m|\infty}$ -module in $\mathcal{O}^{int}(m|\infty)$. Then M is isomorphic to $L(\mathfrak{gl}_{m|\infty}, \Lambda_{m|\infty}(\lambda, \ell))$ for some $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$.*

3.3. The category $\mathcal{O}^{int}(m|n)$. Consider the representations of $\mathfrak{gl}_{m|n}$ of finite rank (see [7, Sections 3.3 and 3.4] for more detail). Let $\mathfrak{l}_{m|n} = \mathfrak{l}_{m|\infty} \cap \mathfrak{gl}_{m|n}$. Let $P_{m|n}^+$ be the set of $\Lambda \in P_{m|\infty}^+$ such that $\langle E_a, \Lambda \rangle = 0$ for $a \notin \mathbb{J}_{m|n}$, and $\mathcal{P}(\mathfrak{g})_{m|n}$ the set of $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$ such that $\Lambda_{m|\infty}(\lambda, \ell) \in P_{m|n}^+$. (We assume that $\mathcal{P}(\mathfrak{g})_{m|\infty} = \mathcal{P}(\mathfrak{g})$.) Let us write $\Lambda_{m|n}(\lambda, \ell) = \Lambda_{m|\infty}(\lambda, \ell)$ for $\Lambda_{m|\infty}(\lambda, \ell) \in P_{m|n}^+$. As in Section 3.1, one may define $L(\mathfrak{l}_{m|n}, \Lambda)$, $K(\mathfrak{gl}_{m|n}, \Lambda)$, $L(\mathfrak{gl}_{m|n}, \Lambda)$ for $\Lambda \in P_{m|n}^+$, and a parabolic category $\mathcal{O}(m|n)$ of $\mathfrak{gl}_{m|n}$ -modules.

Let $M \in \mathcal{O}(m|\infty)$ be given with $M = \bigoplus_{\gamma} M_{\gamma}$. We define the truncation functor $\mathrm{tr}_n : \mathcal{O}(m|\infty) \rightarrow \mathcal{O}(m|n)$ by

$$\mathrm{tr}_n(M) = \bigoplus_{\gamma} M_{\gamma},$$

where the sum is over γ with $\langle E_a, \gamma \rangle = 0$ for $a \notin \mathbb{J}_{m|n}$. For $\Lambda \in P_{m|\infty}^+$, we have $\mathrm{tr}_n(X(\mathfrak{gl}_{m|\infty}, \Lambda)) = X(\mathfrak{gl}_{m|n}, \Lambda)$ if $\Lambda \in P_{m|n}^+$, and 0 otherwise for $X = K, L$ by [7, Lemma 3.2]. Also, it is easy to see that $\mathrm{tr}_n(L(\mathfrak{l}_{m|\infty}, \Lambda)) = L(\mathfrak{l}_{m|n}, \Lambda)$ if $\Lambda \in P_{m|n}^+$, and 0 otherwise.

Definition 3.9. Define $\mathcal{O}^{int}(m|n)$ to be the category of $\mathfrak{gl}_{m|n}$ -modules M satisfying

- (1) $M = \bigoplus_{\gamma \in \mathfrak{h}_{m|n}^*} M_{\gamma}$ and $\dim M_{\gamma} < \infty$ for $\gamma \in \mathfrak{h}_{m|n}^*$,
- (2) $\mathrm{wt}(M) \subset \bigcup_{i=1}^r \left(\ell_i \Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|n}} \mathbb{Z}_{\geq 0} \delta_a \right)$ for some $r \geq 1$ and $\ell_i \in \mathbb{Z}_{\geq 0}$,
- (3) $E_{-\beta_{\overline{m}}}$ is locally nilpotent on M .

As in the case of $\mathcal{O}^{int}(m|\infty)$, each $M \in \mathcal{O}^{int}(m|n)$ is a direct sum of $L(\mathfrak{l}_{m|n}, \Lambda)$'s for $\Lambda \in P_{m|n}^+$ and $\mathcal{O}^{int}(m|n)$ is a full subcategory of $\mathcal{O}(m|n)$.

We will prove that $\mathcal{O}^{int}(m|n)$ is a semisimple tensor category. We remark that the super duality functor is not available to $\mathcal{O}^{int}(m|n)$. So we will prove this in a rather indirect way, but still using super duality.

Lemma 3.10. *Let $V(\mathfrak{g}_{m|n}, \Lambda)$ be a highest weight $\mathfrak{g}_{m|n}$ -module in $\mathcal{O}(m|n)$ with highest weight $\Lambda \in P_{m|n}^+$. Then there exists a highest weight $\mathfrak{g}_{m|\infty}$ -module $V(\mathfrak{g}_{m|\infty}, \Lambda)$ with highest weight Λ in $\mathcal{O}(m|\infty)$ such that $\mathfrak{tr}_n(V(\mathfrak{g}_{m|\infty}, \Lambda)) = V(\mathfrak{g}_{m|n}, \Lambda)$.*

Proof. Note that $V(\mathfrak{g}_{m|n}, \Lambda) \cong K(\mathfrak{g}_{m|n}, \Lambda)/W_n$ for some $\mathfrak{g}_{m|n}$ -submodule W_n of $K(\mathfrak{g}_{m|n}, \Lambda)$. By [27, Lemma 2.1.10], we have a filtration $\{0\} = W_n^{(0)} \subset W_n^{(1)} \subset W_n^{(2)} \subset \dots$ such that

- (1) $W_n = \bigcup_{i \geq 1} W_n^{(i)}$,
- (2) $W_n^{(i)}/W_n^{(i-1)}$ is a highest weight $\mathfrak{g}_{m|n}$ -module with highest weight $\nu_i \in P_{m|n}^+$,
- (3) if $\nu_i - \nu_j \in \sum_{\beta \in \Pi_{m|n}} \mathbb{Z}_{\geq 0} \beta$, then $i < j$,
- (4) given $\gamma \in \text{wt}(W_n)$, $\left(W_n/W_n^{(i)}\right)_\gamma = 0$ for $i \gg 0$.

We use induction to show that there exist $\mathfrak{g}_{m|\infty}$ -submodules $W^{(i)}$ of $K(\mathfrak{g}_{m|\infty}, \Lambda)$ ($i \geq 0$) such that $\{0\} = W^{(0)} \subset W^{(1)} \subset W^{(2)} \subset \dots$ and $\mathfrak{tr}_n(W^{(i)}) = W_n^{(i)}$ for $i \geq 1$.

Suppose that $i = 1$. Let $v_n^{(1)}$ be a $\mathfrak{g}_{m|n}$ -highest weight vector of $W_n^{(1)}$. Since we may regard $K(\mathfrak{g}_{m|n}, \Lambda) \subset K(\mathfrak{g}_{m|\infty}, \Lambda)$ and $K(\mathfrak{g}_{m|\infty}, \Lambda)$ is an integrable $\mathfrak{l}_{m|\infty}$ -module, $v_n^{(1)}$ is also a $\mathfrak{g}_{m|\infty}$ -highest weight vector. We put $W^{(1)} = U(\mathfrak{g}_{m|\infty}^-)v_n^{(1)}$, where $\mathfrak{g}_{m|\infty}^-$ is the subalgebra generated by $E_{-\beta}$ for $\beta \in \Pi_{m|\infty}$. By construction, it is clear that $\mathfrak{tr}_n(W^{(1)}) = W_n^{(1)}$.

Suppose that there exist $W^{(1)} \subset \dots \subset W^{(i-1)}$ such that $\mathfrak{tr}_n(W^{(k)}) = W_n^{(k)}$ for $k = 1, \dots, i-1$. Let $v_n^{(i)}$ be a $\mathfrak{g}_{m|n}$ -highest weight vector of $W_n^{(i)}/W_n^{(i-1)}$. Then by the same argument as above, $v_n^{(i)}$ is also a $\mathfrak{g}_{m|\infty}$ -highest weight vector. Let $W^{(i)}$ be the $\mathfrak{g}_{m|\infty}$ -submodule of $K(\mathfrak{g}_{m|\infty}, \Lambda)$ generated by $W^{(i-1)}$ and $v_n^{(i)}$. Note that $W^{(i)}/W^{(i-1)} = U(\mathfrak{g}_{m|\infty}^-)v_n^{(i)}$ and hence $\mathfrak{tr}_n(W^{(i)}/W^{(i-1)}) = W_n^{(i)}/W_n^{(i-1)}$. By the exactness of \mathfrak{tr}_n and the induction hypothesis, we have $\mathfrak{tr}_n(W^{(i)}) = W_n^{(i)}$. This completes the induction.

Now, if we put $W = \bigcup_{i \geq 1} W^{(i)}$, then $\mathfrak{tr}_n(W) = W_n$. Since $\mathfrak{tr}_n(K(\mathfrak{g}_{m|\infty}, \Lambda)) = K(\mathfrak{g}_{m|n}, \Lambda)$ and \mathfrak{tr}_n is exact, it follows that $\mathfrak{tr}_n(V(\mathfrak{g}_{m|\infty}, \Lambda)) = V(\mathfrak{g}_{m|n}, \Lambda)$, where $V(\mathfrak{g}_{m|\infty}, \Lambda) = K(\mathfrak{g}_{m|\infty}, \Lambda)/W \in \mathcal{O}(m|\infty)$. \square

Theorem 3.11. *Let M be a highest weight $\mathfrak{g}_{m|n}$ -module in $\mathcal{O}^{int}(m|n)$. Then M is isomorphic to $L(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ for some $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$.*

Proof. Let v be a highest weight vector of M with highest weight γ . By (2) in the definition of $\mathcal{O}^{int}(m|n)$ and [9, Theorem 3.27], M is semisimple over $\mathfrak{gl}_{m|n} :=$

$\mathfrak{gl}_{m|\infty} \cap \mathfrak{gl}_{m|n}$ and v is also a highest weight vector of a polynomial module over $\mathfrak{gl}_{m|n}$, which implies that $\gamma = \Lambda_{m|n}(\lambda, \ell)$ for some $\lambda \in \mathcal{P}$ and $\ell \in \mathbb{Z}_{>0}$. Furthermore, we have $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$ by the condition (3) in $\mathcal{O}^{int}(m|n)$.

Suppose that M is not irreducible. Let N be a proper maximal submodule of M . Choose a highest weight vector v' with maximal weight η of N . By the same argument in the previous paragraph, we also have $\eta = \Lambda_{m|n}(\mu, \ell')$ for some $(\mu, \ell') \in \mathcal{P}(\mathfrak{g})_{m|n}$. Since $\eta \in \gamma - \sum_{\beta \in \Pi_{m|n}} \mathbb{Z}_{\geq 0} \beta$, we have $\ell' = \ell$.

Next, by Lemma 3.10, there exists a highest weight module $M_{m|\infty} \in \mathcal{O}(m|\infty)$ with highest weight $\Lambda_{m|\infty}(\lambda, \ell) = \Lambda_{m|n}(\lambda, \ell) \in P_{m|\infty}^+$ such that $\mathfrak{t}_n(M_{m|\infty}) = M$. Here we may assume that $J_{m|\infty} = I_{m|\infty} \setminus \{0\}$. Then by [7, Theorem 4.6 and Proposition 5.3], there exists a highest weight module $M_{m+\infty} \in \mathcal{O}(m+\infty)$ with highest weight $\Lambda_{m+\infty}(\lambda, \ell) \in P_{m+\infty}^+$ such that $\mathcal{F}(M_{m+\infty}) = M_{m|\infty}$.

Note that M is a semisimple $\mathfrak{l}_{m|n}$ -module and v' generates a highest weight $\mathfrak{l}_{m|n}$ -submodule $L(\mathfrak{l}_{m|n}, \Lambda_{m|n}(\mu, \ell))$. Since $M_{m|\infty}$ is a semisimple $\mathfrak{l}_{m|\infty}$ -module and $\mathfrak{t}_n(L(\mathfrak{l}_{m|\infty}, \Lambda_{m|\infty}(\mu, \ell))) = L(\mathfrak{l}_{m|n}, \Lambda_{m|n}(\mu, \ell))$, we conclude that the multiplicity of $L(\mathfrak{l}_{m|\infty}, \Lambda_{m|\infty}(\mu, \ell))$ in $M_{m|\infty}$ is non-zero. This implies that the multiplicity of $L(\mathfrak{l}_{m+\infty}, \Lambda_{m+\infty}(\mu, \ell))$ in $M_{m+\infty}$ is also non-zero by (3.1). In particular, we have $\Lambda_{m+\infty}(\mu, \ell) \in \Lambda_{m+\infty}(\lambda, \ell) - \sum_{\alpha \in \Pi_{m+\infty}} \mathbb{Z}_{\geq 0} \alpha$.

Now, consider the Casimir operator Ω on $M_{m+\infty}$. By [20, Lemma 9.8], we have

$$(\Lambda_{m+\infty}(\lambda, \ell) + 2\rho|\Lambda_{m+\infty}(\lambda, \ell)) = (\Lambda_{m+\infty}(\mu, \ell) + 2\rho|\Lambda_{m+\infty}(\mu, \ell)),$$

where ρ is the Weyl vector for $\mathfrak{gl}_{m+\infty}$. On the other hand, by [20, Lemma 10.3],

$$(\Lambda_{m+\infty}(\lambda, \ell) + 2\rho|\Lambda_{m+\infty}(\lambda, \ell)) > (\Lambda_{m+\infty}(\mu, \ell) + 2\rho|\Lambda_{m+\infty}(\mu, \ell)),$$

which is a contradiction. (The arguments for the case of $\mathfrak{g} = \mathfrak{b}^\bullet$ is almost the same.) Therefore, M is an irreducible highest weight module with highest weight $\Lambda_{m|n}(\lambda, \ell)$. \square

For $M \in \mathcal{O}(m|n)$, let $M^* = \bigoplus_{\gamma} M_{\gamma}^*$, where $M_{\gamma}^* = \text{Hom}_{\mathbb{C}}(M_{\gamma}, \mathbb{C})$. We define a $\mathfrak{gl}_{m|n}$ -module structure on M^* by $\langle m, uf \rangle = (-1)^{|m||u|+1} \langle um, f \rangle$, for $f \in M^*$ and homogeneous elements $u \in \mathfrak{gl}_{m|n}$, $m \in M$. Note that $(M^*)^* \cong M$ and M^* is a lowest weight module with lowest weight $-\Lambda$ if M is a highest weight module with highest weight Λ . In particular, $L(\mathfrak{gl}_{m|n}, \Lambda)^*$ is the irreducible lowest weight module with lowest weight $-\Lambda$ for $\Lambda \in P_{m|n}^+$.

Theorem 3.12. *$\mathcal{O}^{int}(m|n)$ is a semisimple tensor category, whose irreducible objects are $L(\mathfrak{gl}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$.*

Proof. Suppose that $M \in \mathcal{O}^{int}(m|n)$ is given. Let v be a maximal weight vector with highest weight μ , and let $N = U(\mathfrak{g}_{m|n})v = U(\mathfrak{g}_{m|n}^-)v$. By Theorem 3.11, we have $\mu = \Lambda_{m|n}(\lambda, \ell)$ for some $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$ and $N \cong L(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$.

Let $v^* \in M_\mu^*$ be such that $\langle v, v^* \rangle = 1$, and let $L = U(\mathfrak{g}_{m|n})v^* = U(\mathfrak{g}_{m|n}^+)v^* \subset M^*$, which is a lowest weight module with lowest weight $-\Lambda_{m|n}(\lambda, \ell)$. Then $L^* \in \mathcal{O}^{int}(m|n)$ and hence isomorphic to $L(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ by Theorem 3.11. By taking dual of the embedding $L \hookrightarrow M^*$ and then composing with $N \hookrightarrow M$, we have

$$(3.6) \quad N \longrightarrow M \cong (M^*)^* \longrightarrow L^*.$$

Since v maps to a non-zero vector in L^* and $L^* \cong N$, (3.6) gives an isomorphism of N onto itself by Schur's lemma, which implies that the short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ splits and $M \cong N \oplus M/N$. Therefore, M is completely reducible.

Finally, it is clear that $\mathcal{O}^{int}(m|n)$ is closed under tensor product. This completes the proof. \square

Remark 3.13. The arguments for Theorems 3.11 and 3.12 are also available when $n = \infty$, which give alternative proofs of Theorem 3.7 and Corollary 3.8.

4. CATEGORY $\mathcal{O}_q^{int}(m|n)$ OVER THE QUANTUM SUPERALGEBRA $U_q(\mathfrak{g}_{m|n})$

In this section, we consider the q -analogue of a module in $\mathcal{O}^{int}(m|n)$ over the quantized enveloping algebra $U_q(\mathfrak{g}_{m|n})$, and prove its semisimplicity.

4.1. The quantum superalgebra $U_q(\mathfrak{g}_{m|n})$. From now on, we assume that $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Let $A = (a_{ij}) = (\langle \beta_i^\vee, \beta_j \rangle)_{i,j \in I_{m|n}}$ the generalized Cartan matrix for $\mathfrak{g}_{m|n}$. Let

$$P_{m|n} = \bigoplus_{a \in \mathbb{J}_{m|n}} \mathbb{Z}\delta_a \oplus \mathbb{Z}\Lambda_{\overline{m}}, \quad P_{m|n}^\vee = \bigoplus_{a \in \mathbb{J}_{m|n}} \mathbb{Z}E_a \oplus \mathbb{Z}rK$$

be the weight lattice and dual weight lattice, respectively, where $r = 1$ for $\mathfrak{g} = \mathfrak{c}$ and $r = 2$ otherwise.

Let q be an indeterminate. Put $q_i = q^{\overline{s}_i}$ for $i \in I_{m|n}$, and

$$[r]_i = \frac{q_i^r - q_i^{-r}}{q_i - q_i^{-1}}, \quad [r]_i! = \prod_{k=1}^r [k]_i,$$

for $r \geq 0$, where $\overline{s}_i = -s_i$ (see Remarks 5.1 and 8.1 for the difference when we use \overline{s}_i instead of s_i (2.3)).

The quantum superalgebra $U_q(\mathfrak{g}_{m|n})$ is the associative superalgebra (or \mathbb{Z}_2 -graded algebra) with 1 over $\mathbb{Q}(q)$ generated by e_i, f_i ($i \in I_{m|n}$) and q^h ($h \in P_{m|n}^\vee$), which

are subject to the following relations [33]:

$$\begin{aligned}
& \deg(q^h) = 0, \quad \deg(e_i) = \deg(f_i) = |\beta_i|, \\
& q^0 = 1, \quad q^{h+h'} = q^h q^{h'}, \\
& q^h e_i = q^{\langle h, \beta_i \rangle} e_i q^h, \quad q^h f_i = q^{-\langle h, \beta_i \rangle} f_i q^h, \\
& e_i f_j - (-1)^{|\beta_i||\beta_j|} f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \\
& z_i z_j - (-1)^{|\beta_i||\beta_j|} z_j z_i = 0, \quad \text{if } a_{ij} = 0, \\
& \sum_{r=0}^{1+|a_{ij}|} (-1)^r z_i^{(r)} z_j z_i^{(1+|a_{ij}|-r)} = 0, \quad \text{if } i \neq 0 \text{ and } a_{ij} \neq 0, \\
& z_0 z_{\overline{1}} z_0 z_{\frac{1}{2}} + z_{\overline{1}} z_0 z_{\frac{1}{2}} z_0 + z_0 z_{\frac{1}{2}} z_0 z_{\overline{1}} + z_{\frac{1}{2}} z_0 z_{\overline{1}} z_0 - (q + q^{-1}) z_0 z_{\overline{1}} z_{\frac{1}{2}} z_0 = 0,
\end{aligned}$$

for $i, j \in I_{m|n}$, $h, h' \in P_{m|n}^\vee$ and $z = e, f$, where $t_i = q^{\overline{s}_i \beta_i^\vee}$ and $z_i^{(r)} = \frac{z_i^r}{[r]_i!}$ for $r \geq 0$.

Let U_q^+ (resp. U_q^-) be the subalgebra of $U_q(\mathfrak{g}_{m|n})$ generated by e_i (resp. f_i) for $i \in I_{m|n}$ and U_q^0 the subalgebra generated by q^h for $h \in P_{m|n}^\vee$. We have triangular decomposition $U_q(\mathfrak{g}_{m|n}) \cong U_q^- \otimes U_q^0 \otimes U_q^+$ as a $\mathbb{Q}(q)$ -space. There is a Hopf superalgebra structure on $U_q(\mathfrak{g}_{m|n})$, where the comultiplication Δ is given by

$$\begin{aligned}
\Delta(q^h) &= q^h \otimes q^h, \\
\Delta(e_i) &= e_i \otimes t_i^{-1} + 1 \otimes e_i, \\
\Delta(f_i) &= f_i \otimes 1 + t_i \otimes f_i,
\end{aligned}$$

the antipode S is given by $S(q^h) = q^{-h}$, $S(e_i) = -e_i t_i$, $S(f_i) = -t_i^{-1} f_i$, and the counit ε is given by $\varepsilon(q^h) = 1$, $\varepsilon(e_i) = \varepsilon(f_i) = 0$ for $h \in P_{m|n}^\vee$ and $i \in I_{m|n}$. We will also need the following subalgebras of $U_q(\mathfrak{g}_{m|n})$:

$$\begin{aligned}
U_q(\mathfrak{gl}_{m|n}) &= \langle e_i, f_i, q^{\pm E_a} \mid i \in I_{m|n} \setminus \{\overline{m}\}, a \in \mathbb{J}_{m|n} \rangle, \\
U_q(\mathfrak{gl}_{m|0}) &= \langle e_i, f_i, q^{\pm E_a} \mid i \in I_{m|0} \setminus \{\overline{m}\}, a \in (\mathbb{J}_{m|n})_0 \rangle, \\
U_q(\mathfrak{gl}_{0|n}) &= \langle e_i, f_i, q^{\pm E_a} \mid i \in I_{0|n}, a \in (\mathbb{J}_{m|n})_1 \rangle,
\end{aligned}$$

where $I_{m|0} = \{\overline{m}, \dots, \overline{1}\}$ and $I_{0|n} = I_{m|n} \setminus \{\overline{m}, \dots, \overline{1}, 0\}$.

4.2. Classical limit. We can define the notion of a highest weight $U_q(\mathfrak{g}_{m|n})$ -module by the triangular decomposition of $U_q(\mathfrak{g}_{m|n})$, and consider its classical limit in the same way as in symmetrizable Kac-Moody algebras. We leave the detailed verification to the reader (see [16, 29]).

For a $U_q(\mathfrak{g}_{m|n})$ -module M and $\gamma \in P_{m|n}$, let $M_\gamma = \{m \mid q^h m = q^{\langle h, \mu \rangle} m \text{ (} h \in P_{m|n}^\vee \text{)}\} \subset M$ and $\text{wt}(M) = \{\gamma \in P_{m|n} \mid M_\gamma \neq 0\}$. Suppose that M is a $U_q(\mathfrak{g}_{m|n})$ -module generated by a highest weight vector u of weight $\Lambda \in P_{m|n}$. Then $M = \bigoplus_\mu M_\mu$, where the sum is over $\mu \in \Lambda - \sum_{\beta \in \Pi_{m|n}} \mathbb{Z}_{\geq 0} \beta$.

Let $\mathbf{A} = \mathbb{Q}[q, q^{-1}]$. Let $M_{\mathbf{A}}$ be the \mathbf{A} -span of $f_{i_1} \dots f_{i_r} u$ for $r \geq 0$ and $i_1, \dots, i_r \in I_{m|n}$, and $M_{\mu, \mathbf{A}} = M_{\mathbf{A}} \cap M_\mu$. Then $M_{\mathbf{A}} = \bigoplus_\mu M_{\mu, \mathbf{A}}$, and $\text{rank}_{\mathbf{A}} M_{\mu, \mathbf{A}} = \dim_{\mathbb{Q}(q)} M_\mu$. One can check that the \mathbf{A} -module $M_{\mathbf{A}}$ is invariant under e_i , f_i , q^h and $\frac{q^h - q^{-h}}{q - q^{-1}}$ for $i \in I_{m|n}$ and $h \in P_{m|n}^\vee$. Set $\overline{M} = M_{\mathbf{A}} \otimes_{\mathbf{A}} \mathbb{C}$ and $\overline{M}_\mu = M_{\mu, \mathbf{A}} \otimes_{\mathbf{A}} \mathbb{C}$. Here \mathbb{C} is understood to be an \mathbf{A} -module where $f(x) \cdot c = f(1)c$ for $f(x) \in \mathbf{A}$ and $c \in \mathbb{C}$. We have $\overline{M} = \bigoplus_\mu \overline{M}_\mu$ with $\dim_{\mathbb{C}} \overline{M}_\mu = \text{rank}_{\mathbf{A}} M_{\mu, \mathbf{A}}$.

Recall that the enveloping algebra $U(\mathfrak{g}_{m|n})$ is isomorphic to the associative superalgebra with 1 over \mathbb{C} generated by x_i^\pm ($i \in I_{m|n}$) and $h \in P_{m|n}^\vee$ subject to the following relations [33, Theorem 10.5.8]:

$$\begin{aligned} \deg(h) &= 0, \quad \deg(x_i^\pm) = |\beta_i|, \\ [h, h'] &= 0, \quad [h, x_i^\pm] = \pm \langle h, \beta_i \rangle x_i^\pm, \quad [x_i^+, x_j^-] = \delta_{ij} \beta_i^\vee, \\ [z_i, z_j] &= 0, \text{ if } a_{ij} = 0, \quad \sum_{r=0}^{1+|a_{ij}|} (-1)^r z_i^{(r)} z_j z_i^{(1+|a_{ij}|-r)} = 0, \text{ if } i \neq 0 \text{ and } a_{ij} \neq 0, \\ z_0 z_{\overline{1}} z_0 z_{\frac{1}{2}} + z_{\overline{1}} z_0 z_{\frac{1}{2}} z_0 + z_0 z_{\frac{1}{2}} z_0 z_{\overline{1}} + z_{\frac{1}{2}} z_0 z_{\overline{1}} z_0 - 2 z_0 z_{\overline{1}} z_{\frac{1}{2}} z_0 &= 0, \end{aligned}$$

for $i, j \in I_{m|n}$ and $h, h' \in P_{m|n}^\vee$, where $z = x^\pm$ and $z^{(r)} = \frac{z^r}{r!}$ for $r \geq 0$. Here $[\ , \]$ denotes the superbracket $[u, v] = uv - (-1)^{|u||v|}vu$ for homogeneous elements $u, v \in U(\mathfrak{g}_{m|n})$.

Let \overline{e}_i , \overline{f}_i and \overline{h} be the \mathbb{C} -linear endomorphisms on \overline{M} induced from e_i , f_i and $\frac{q^h - q^{-h}}{q - q^{-1}}$ for $i \in I_{m|n}$ and $h \in P_{m|n}^\vee$, and $\overline{U}_{\overline{M}}$ the subalgebra of $\text{End}_{\mathbb{C}}(\overline{M})$ generated by them. Then there exists a \mathbb{C} -algebra homomorphism from $U(\mathfrak{g}_{m|n})$ to $\overline{U}_{\overline{M}}$ sending x_i^+ , x_i^- and h to \overline{e}_i , \overline{f}_i and \overline{h} , respectively. Hence, \overline{M} is a $U(\mathfrak{g}_{m|n})$ -module with highest weight Λ , which is called the classical limit of M .

4.3. The category $\mathcal{O}_q^{\text{int}}(m|n)$.

Definition 4.1. Let $\mathcal{O}_q^{\text{int}}(m|n)$ be the category of $U_q(\mathfrak{g}_{m|n})$ -modules M satisfying

- (1) $M = \bigoplus_{\gamma \in P_{m|n}} M_\gamma$ and $\dim M_\gamma < \infty$ for $\gamma \in P_{m|n}$,
- (2) $\text{wt}(M) \subset \bigcup_{i=1}^r \left(\ell_i \Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|n}} \mathbb{Z}_{\geq 0} \delta_a \right)$ for some $r \geq 1$ and $\ell_i \in \mathbb{Z}_{\geq 0}$,
- (3) $f_{\overline{m}}$ is locally nilpotent on M .

For $\Lambda \in P_{m|n}$, let $L_q(\mathfrak{g}_{m|n}, \Lambda)$ denote the irreducible highest weight $U_q(\mathfrak{g}_{m|n})$ -module with highest weight Λ .

Theorem 4.2. *Let M be a highest weight $U_q(\mathfrak{g}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$. Then M is isomorphic to $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ for some $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$.*

Proof. By Section 4.2, the classical limit \overline{M} is a highest weight module in $\mathcal{O}^{int}(m|n)$. By Corollary 3.8 and Theorem 3.11, \overline{M} is isomorphic to $L(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ for some $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$. Since $\dim_{\mathbb{Q}(q)} M_\gamma = \dim_{\mathbb{C}} \overline{M}_\gamma$ for $\gamma \in P_{m|n}$, this forces M to be an irreducible highest weight module with highest weight $\Lambda_{m|n}(\lambda, \ell)$, that is, $M \cong L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$. \square

Let $M \in \mathcal{O}_q^{int}(m|n)$ be given. Let $M^* = \bigoplus_{\lambda \in P} M_\lambda^*$ with $M_\lambda^* = \text{Hom}_{\mathbb{Q}(q)}(M_\lambda, \mathbb{Q}(q))$. Define a $U_q(\mathfrak{g}_{m|n})$ -module structure on M^* by $\langle m, uf \rangle = (-1)^{|u||m|} \langle S(u)m, f \rangle$ for $f \in M^*$ and homogeneous elements $u \in U_q(\mathfrak{g}_{m|n})$ and $m \in M$. Also, let $M^{*'}$ be another $U_q(\mathfrak{g}_{m|n})$ -module with the same underlying vector space as M^* , where the action is given by $\langle m, uf \rangle = (-1)^{|u||m|} \langle S^{-1}(u)m, f \rangle$. By definition, we have $(M^*)^{*'} \cong M \cong (M^{*'})^*$.

Let $N = M^\diamond$ ($\diamond = *', *$). Then N is a lowest weight module with lowest weight $-\Lambda$ if M is a highest weight U_q -module with highest weight $\Lambda \in P_{m|n}$. In particular, we have $L_q(\mathfrak{g}_{m|n}, \Lambda)^\diamond$ ($\diamond = *', *$) is the irreducible lowest weight module with lowest weight $-\Lambda$.

Theorem 4.3. *$\mathcal{O}_q^{int}(m|n)$ is a semisimple tensor category, whose irreducible objects are $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$.*

Proof. The arguments are almost identical to those in Theorem 3.12. Let $M \in \mathcal{O}_q^{int}(m|n)$ be given and let N be a submodule generated by a maximal weight vector v with highest weight μ . By Theorem 4.2, we have $\mu = \Lambda_{m|n}(\lambda, \ell)$ for some $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$ and $N \cong L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$.

Let $v^* \in M_\mu^*$ be such that $\langle v, v^* \rangle = 1$, and let L be lowest weight submodule of M^* generated by v^* with lowest weight $-\Lambda_{m|n}(\lambda, \ell)$. Then $L^{*'} \in \mathcal{O}_q^{int}(m|n)$ and it is isomorphic to $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ by Theorem 4.2. As in (3.6), the dual of the embedding $L \hookrightarrow M^*$ with respect to $*'$ yields a left inverse of the embedding $N \hookrightarrow M$, which implies that $M \cong N \oplus M/N$ and hence M is semisimple.

It is clear that $\mathcal{O}_q^{int}(m|n)$ is closed under tensor product, and therefore it is a semisimple tensor category. \square

5. CRYSTAL BASE OF A q -DEFORMED FOCK SPACE

In this section, we recall the crystal base theory for contragredient Lie superalgebras [2, Section 2.3], and consider a crystal base of a q -deformed Fock space \mathcal{V}_q over $U_q(\mathfrak{g}_{m|n})$.

5.1. Crystal bases. Let M be a $U_q(\mathfrak{g}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$.

First, suppose that $i \in I_{m|n} \setminus \{0\}$ is given, that is, β_i is an even simple root. For $u \in M$ of weight λ , we have a unique expression

$$u = \sum_{k \geq 0, -\langle \beta_i^\vee, \lambda \rangle} f_i^{(k)} u_k,$$

where $e_i u_k = 0$ for all $k \geq 0$. Then we define the Kashiwara operators \tilde{e}_i and \tilde{f}_i as follows:

(1) For $i \in I_{m|0}$,

$$\tilde{e}_i u = \sum_k q_i^{l_k - 2k + 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_k q_i^{-l_k + 2k + 1} f_i^{(k+1)} u_k,$$

where $l_k = \langle \beta_i^\vee, \lambda + k\beta_i \rangle$ for $k \geq 0$.

(2) For $i \in I_{0|n}$,

$$\tilde{e}_i u = \sum_k f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_k f_i^{(k+1)} u_k.$$

Next, suppose that $i = 0$, that is, β_0 is an odd isotropic simple root. We define

$$\tilde{e}_0 u = e_0 u, \quad \tilde{f}_0 u = q_0 f_0 t_0^{-1} u.$$

Now, let \mathbb{A} denote the subring of $\mathbb{Q}(q)$ consisting of all rational functions which are regular at $q = 0$. Then a pair (L, B) is called a *crystal base of M* if

- (1) L is an \mathbb{A} -lattice of M , where $L = \bigoplus_{\lambda \in P_{m|n}} L_\lambda$ with $L_\lambda = L \cap M_\lambda$,
- (2) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for $i \in I_{m|n}$,
- (3) B is a pseudo-basis of L/qL (i.e. $B = B^\bullet \cup (-B^\bullet)$ for a \mathbb{Q} -basis B^\bullet of L/qL),
- (4) $B = \bigsqcup_{\lambda \in P_{m|n}} B_\lambda$ with $B_\lambda = B \cap (L/qL)_\lambda$,
- (5) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$ for $i \in I_{m|n}$,
- (6) for $b, b' \in B$ and $k \in I_{m|n}$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

The set $B/\{\pm 1\}$ has an $I_{m|n}$ -colored oriented graph structure, where $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i b = b'$ for $i \in I_{m|n}$ and $b, b' \in B/\{\pm 1\}$. We call $B/\{\pm 1\}$ the *crystal of M* . For $b \in B$ and $i \in I_{m|n}$, we set $\varepsilon_i(b) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r b \neq 0\}$ and $\varphi_i(b) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^r b \neq 0\}$. We denote the weight of b by $\text{wt}(b)$.

Remark 5.1. A crystal base (L, B) of a $U_q(\mathfrak{g}_{m|n})$ -module is also a crystal base as a $U_q(\mathfrak{gl}_{m|n})$ -module in the sense of [2], which in particular implies that it is a upper crystal base of M as a $U_q(\mathfrak{gl}_{m|0})$ -module, and a lower crystal base of M as a $U_q(\mathfrak{gl}_{0|n})$ -module (see [2, Lemma 2.5]).

Let M_i ($i = 1, 2$) be a $U_q(\mathfrak{g}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$ with a crystal base (L_i, B_i) . Then $(L_1 \otimes L_2, B_1 \otimes B_2)$ is a crystal base of $M_1 \otimes M_2$ [2, Proposition 2.8]. The actions of \tilde{e}_i and \tilde{f}_i on $B_1 \otimes B_2$ are as follows.

For $i \in I_{0|n}$, we have

$$(5.1) \quad \begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} (\tilde{e}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes (\tilde{e}_i b_2), & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} (\tilde{f}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes (\tilde{f}_i b_2), & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

For $i \in I_{m|0}$, we have

$$(5.2) \quad \begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes (\tilde{e}_i b_2), & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ (\tilde{e}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes (\tilde{f}_i b_2), & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ (\tilde{f}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1). \end{cases} \end{aligned}$$

For $i = 0$, we have

$$(5.3) \quad \begin{aligned} \tilde{e}_0(b_1 \otimes b_2) &= \begin{cases} \pm b_1 \otimes (\tilde{e}_0 b_2), & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle > 0, \\ (\tilde{e}_0 b_1) \otimes b_2, & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle = 0, \end{cases} \\ \tilde{f}_0(b_1 \otimes b_2) &= \begin{cases} \pm b_1 \otimes (\tilde{f}_0 b_2), & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle > 0, \\ (\tilde{f}_0 b_1) \otimes b_2, & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle = 0. \end{cases} \end{aligned}$$

5.2. q -deformed Clifford-Weyl algebra. Let \mathcal{A}_q be an associative $\mathbb{Q}(q)$ -algebra with 1 generated by $\psi_a, \psi_a^*, \omega_a$ and ω_a^{-1} for $a \in \pm \mathbb{J}_{m|n}$ subject to the following relations:

$$\begin{aligned} \omega_a \omega_b &= \omega_b \omega_a, \quad \omega_a \omega_a^{-1} = 1, \\ \omega_a \psi_b \omega_a^{-1} &= q^{(-1)^{|a|} \delta_{ab}} \psi_b, \quad \omega_a \psi_b^* \omega_a^{-1} = q^{-(-1)^{|a|} \delta_{ab}} \psi_b^*, \\ \psi_a \psi_b + (-1)^{|a||b|} \psi_b \psi_a &= 0, \quad \psi_a^* \psi_b^* + (-1)^{|a||b|} \psi_b^* \psi_a^* = 0, \\ \psi_a \psi_b^* + (-1)^{|a||b|} \psi_b^* \psi_a &= 0 \quad (a \neq b), \\ \psi_a \psi_a^* &= [q \omega_a], \quad \psi_a^* \psi_a = (-1)^{1+|a|} [\omega_a], \end{aligned}$$

Here $[q^k \omega_a^{\pm 1}] = \frac{q^k \omega_a^{\pm 1} - q^{-k} \omega_a^{\mp 1}}{q - q^{-1}}$ for $k \in \mathbb{Z}$ and $a \in \pm \mathbb{J}_{m|n}$. Note that the subalgebra generated by ψ_a, ψ_a^* and $\omega_a^{\pm 1}$ for $a \in \pm (\mathbb{J}_{m|n})_0$ (resp. $\pm (\mathbb{J}_{m|n})_1$) is a q -deformed

Clifford algebra (resp. q -deformed Weyl algebra) introduced by Hayashi [11]. Let \mathcal{A}_q^- (resp. \mathcal{A}_q^+) be the subalgebra generated by ψ_a, ψ_a^* for $a \in -\mathbb{J}_{m|n}$ (resp. $a \in \mathbb{J}_{m|n}$).

Let \mathcal{F}_q be the \mathcal{A}_q -module generated by $|0\rangle$ satisfying

$$\psi_{-a}|0\rangle = \psi_b^*|0\rangle = 0, \quad \omega_{-a}|0\rangle = |0\rangle, \quad \omega_b|0\rangle = q^{-1}|0\rangle$$

for $a, b \in \mathbb{J}_{m|n}$. Let \mathcal{F}_q^- (resp. \mathcal{F}_q^+) be the \mathcal{A}_q^- -submodule (resp. \mathcal{A}_q^+ -submodule) of \mathcal{F}_q generated by $|0\rangle$.

Let \mathbf{B} be the set of sequences $\mathbf{m} = (m_a)$ of non-negative integers indexed by $\mathbb{J}_{m|n} \cup (-\mathbb{J}_{m|n})$ such that $m_a \leq 1$ for $|a| = 1$. For $\mathbf{m} = (m_a) \in \mathbf{B}$, let

$$\psi_{\mathbf{m}} = \overrightarrow{\prod_{a \in -\mathbb{J}_{m|n}} \psi_a^{*(m_a)}} \overrightarrow{\prod_{b \in \mathbb{J}_{m|n}} \psi_b^{(m_b)}},$$

where the product is taken in the order of $<$ on $\bar{\mathbb{J}}_m$ and

$$\psi_a^{(r)} = \frac{(\psi_a)^r}{[r]!}, \quad \psi_a^{*(r)} = \frac{(\psi_a^*)^r}{[r]!},$$

with $[r]! = [r] \dots [1]$ and $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ for $k, r \geq 0$. By similar arguments as in [11, Proposition 2.1], we can check that \mathcal{F}_q is an irreducible \mathcal{A}_q -module with a $\mathbb{Q}(q)$ -linear basis $\{\psi_{\mathbf{m}}|0\rangle \mid \mathbf{m} \in \mathbf{B}\}$.

Let \mathbf{B}^- (resp. \mathbf{B}^+) be the set of $\mathbf{m} = (m_a) \in \mathbf{B}$ such that $m_a = 0$ for $a \in \mathbb{J}_{m|n}$ (resp. $a \in -\mathbb{J}_{m|n}$). Then $\{\psi_{\mathbf{m}}|0\rangle \mid \mathbf{m} \in \mathbf{B}^-\}$ and $\{\psi_{\mathbf{m}}|0\rangle \mid \mathbf{m} \in \mathbf{B}^+\}$ are $\mathbb{Q}(q)$ -bases of \mathcal{F}_q^- and \mathcal{F}_q^+ , respectively.

Let us describe an action of $U_q(\mathfrak{gl}_{m|n}) \subset U_q(\mathfrak{gl}_{m|n})$ on \mathcal{F}_q . For $i \in I_{m|n} \setminus \{\bar{m}\}$, put

$$\begin{aligned} \mathbf{t}_i^+ &= \begin{cases} \omega_{\frac{-1}{k+1}} \omega_{\bar{k}}, & \text{if } i = \bar{k} \in I_{m|0}, \\ \omega_{\bar{1}}^{-1} \omega_{\frac{1}{2}}, & \text{if } i = 0, \\ \omega_i^{-1} \omega_{i+1}, & \text{if } i \in I_{0|n}, \end{cases} & \mathbf{t}_i^- &= \begin{cases} \omega_{\frac{-1}{k}} \omega_{-\bar{k}+1}, & \text{if } i = \bar{k} \in I_{m|0}, \\ \omega_{\frac{-1}{2}} \omega_{-\bar{1}}, & \text{if } i = 0, \\ \omega_{-i-1}^{-1} \omega_{-i}, & \text{if } i \in I_{0|n}, \end{cases} \\ \mathbf{e}_i^+ &= \begin{cases} \psi_{\frac{-1}{k+1}} \psi_{\bar{k}}^*, & \text{if } i = \bar{k} \in I_{m|0}, \\ \psi_{\bar{1}}^{-1} \psi_{\frac{1}{2}}^*, & \text{if } i = 0, \\ \psi_i \psi_{i+1}^*, & \text{if } i \in I_{0|n}, \end{cases} & \mathbf{e}_i^- &= \begin{cases} \psi_{-\bar{k}} \psi_{-\bar{k}+1}^*, & \text{if } i = \bar{k} \in I_{m|0}, \\ \psi_{-\frac{1}{2}} \psi_{-\bar{1}}^*, & \text{if } i = 0, \\ \psi_{-i-1} \psi_{-i}^*, & \text{if } i \in I_{0|n}, \end{cases} \\ \mathbf{f}_i^+ &= \begin{cases} \psi_{\bar{k}} \psi_{\frac{1}{k+1}}^*, & \text{if } i = \bar{k} \in I_{m|0}, \\ -\psi_{\frac{1}{2}} \psi_{\bar{1}}^*, & \text{if } i = 0, \\ \psi_{i+1} \psi_i^*, & \text{if } i \in I_{0|n}, \end{cases} & \mathbf{f}_i^- &= \begin{cases} \psi_{-\bar{k}+1} \psi_{-\bar{k}}^*, & \text{if } i = \bar{k} \in I_{m|0}, \\ \psi_{-\bar{1}} \psi_{-\frac{1}{2}}^*, & \text{if } i = 0, \\ \psi_{-i} \psi_{-i-1}^*, & \text{if } i \in I_{0|n}. \end{cases} \end{aligned}$$

Since \mathcal{A}_q itself is an \mathcal{A}_q -module under left multiplication, we may regard \mathbf{t}_i^\pm , \mathbf{e}_i^\pm and \mathbf{f}_i^\pm as $\mathbb{Q}(q)$ -linear operators on \mathcal{F}_q under left multiplication.

Lemma 5.2. $\mathcal{F}_{q^r}^\pm$ has a $U_q(\mathfrak{gl}_{m|n})$ -module structure $\rho^\pm : U_q(\mathfrak{gl}_{m|n}) \longrightarrow \text{End}_{\mathbb{Q}(q)}(\mathcal{F}_{q^r}^\pm)$ such that

$$\rho^+(q^{\pm E_a}) = \begin{cases} q^{\pm 1} \omega_a^{\pm \frac{1}{r}}, & \text{if } |a| = 0, \\ q^{\mp 1} \omega_a^{\mp \frac{1}{r}}, & \text{if } |a| = 1, \end{cases} \quad \rho^-(q^{\pm E_a}) = \begin{cases} \omega_{-a}^{\mp \frac{1}{r}}, & \text{if } |a| = 0, \\ \omega_{-a}^{\pm \frac{1}{r}}, & \text{if } |a| = 1, \end{cases}$$

$$\rho^\pm(e_i) = \mathbf{e}_i^\pm, \quad \rho^\pm(f_i) = \mathbf{f}_i^\pm,$$

for $a \in \mathbb{J}_{m|n}$ and $i \in I_{m|n} \setminus \{\overline{m}\}$. Here $r = 2$ when $\mathfrak{g}_{m|n} = \mathfrak{b}_{m|n}$, and $r = 1$ otherwise.

We understand $\omega_a^{\pm \frac{1}{2}}$ as an operator given by $(\omega_a^{\pm 1} v)^{\frac{1}{2}}$ for $v \in \mathcal{F}_{q^2}^\pm$.

Proof. Put $\rho^\pm(t_i) = \mathbf{t}_i^\pm$ for $i \in I_{m|n} \setminus \{\overline{m}\}$. We can check that for $i, j \in I_{m|n} \setminus \{\overline{m}\}$ and $h \in \bigoplus_{a \in \mathbb{J}_{m|n}} \mathbb{Z} E_a$

$$\mathbf{e}_i^\pm \mathbf{f}_j^\pm - (-1)^{|\beta_i||\beta_j|} \mathbf{f}_j^\pm \mathbf{e}_i^\pm = \delta_{ij} \frac{\mathbf{t}_i^\pm - (\mathbf{t}_i^\pm)^{-1}}{q_i - q_i^{-1}},$$

$$\mathbf{e}_i^\pm \mathbf{e}_j^\pm - (-1)^{|\beta_i||\beta_j|} \mathbf{e}_j^\pm \mathbf{e}_i^\pm = \mathbf{f}_i^\pm \mathbf{f}_j^\pm - (-1)^{|\beta_i||\beta_j|} \mathbf{f}_j^\pm \mathbf{f}_i^\pm = 0 \quad \text{if } a_{ij} = 0,$$

$$\rho^\pm(q^h) \mathbf{e}_i^\pm \rho^\pm(q^{-h}) = q^{\langle h, \beta_i \rangle} \mathbf{e}_i^\pm, \quad \rho^\pm(q^h) \mathbf{f}_i^\pm \rho^\pm(q^{-h}) = q^{-\langle h, \beta_i \rangle} \mathbf{f}_i^\pm$$

(cf. [11, Lemma 3.1]). First, we see that $\mathcal{F}_{q^r}^\pm$ is a $U_q(\mathfrak{gl}_{m|0}) \oplus U_q(\mathfrak{gl}_{0|n})$ -module by [25, Proposition B.1] since e_i, f_i are locally nilpotent on $\mathcal{F}_{q^r}^\pm$ for $i \in I_{m|n} \setminus \{\overline{m}, 0\}$. Also, it is straightforward to check that \mathbf{e}_i^\pm and \mathbf{f}_i^\pm ($i = \overline{1}, 0, 1$) satisfy the other relevant relations in $U_q(\mathfrak{gl}_{m|n})$. This implies that $\mathcal{F}_{q^r}^\pm$ is a $U_q(\mathfrak{gl}_{m|n})$ -module. \square

5.3. Crystal bases of q -deformed Fock spaces. Let us put

$$(5.4) \quad \mathcal{V}_q = \begin{cases} \mathcal{F}_q, & \text{if } \mathfrak{g} = \mathfrak{c}, \\ \mathcal{F}_{q^2}^+, & \text{if } \mathfrak{g} = \mathfrak{b}, \\ \mathcal{F}_q^+, & \text{if } \mathfrak{g} = \mathfrak{d}, \\ \mathcal{F}_{q^2}^+ \otimes \mathcal{F}_{q^2}^+, & \text{if } \mathfrak{g} = \mathfrak{b}^\bullet. \end{cases}$$

Proposition 5.3. \mathcal{V}_q has a $U_q(\mathfrak{gl}_{m|n})$ -module structure $\rho : U_q(\mathfrak{gl}_{m|n}) \longrightarrow \text{End}_{\mathbb{Q}(q)}(\mathcal{V}_q)$ as follows: for $a \in \mathbb{J}_{m|n}$ and $i \in I_{m|n}$,

(1) if $\mathfrak{g} = \mathfrak{c}$, then

$$\rho(q^{\pm E_a}) = \rho^+(q^{\pm E_a}) \rho^-(q^{\pm E_a}), \quad \rho(q^K) = q,$$

$$\rho(e_i) = \begin{cases} \psi_{-\overline{m}} \psi_{\overline{m}}^*, & \text{if } i = \overline{m}, \\ \mathbf{e}_i^- (\mathbf{t}_i^+)^{-1} + \mathbf{e}_i^+, & \text{if } i \neq \overline{m}, \end{cases} \quad \rho(f_i) = \begin{cases} \psi_{\overline{m}} \psi_{-\overline{m}}^*, & \text{if } i = \overline{m}, \\ \mathbf{f}_i^- + \mathbf{t}_i^- \mathbf{f}_i^+, & \text{if } i \neq \overline{m}, \end{cases}$$

(2) if $\mathfrak{g} = \mathfrak{b}$, then

$$\rho(q^{\pm E_a}) = \rho^+(q^{\pm E_a}), \quad \rho(q^{2K}) = q,$$

$$\rho(e_i) = \begin{cases} \psi_{\overline{m}}^*, & \text{if } i = \overline{m}, \\ \mathbf{e}_i^+, & \text{if } i \neq \overline{m}, \end{cases} \quad \rho(f_i) = \begin{cases} \psi_{\overline{m}}, & \text{if } i = \overline{m}, \\ \mathbf{f}_i^+, & \text{if } i \neq \overline{m}, \end{cases}$$

(3) if $\mathfrak{g} = \mathfrak{d}$, then

$$\rho(q^{\pm E_a}) = \rho^+(q^{\pm E_a}), \quad \rho(q^{2K}) = q,$$

$$\rho(e_i) = \begin{cases} \psi_{\overline{m}}^* \psi_{\overline{m}-1}^*, & \text{if } i = \overline{m}, \\ \mathbf{e}_i^+, & \text{if } i \neq \overline{m}, \end{cases} \quad \rho(f_i) = \begin{cases} -\psi_{\overline{m}} \psi_{\overline{m}-1}, & \text{if } i = \overline{m}, \\ \mathbf{f}_i^+, & \text{if } i \neq \overline{m}, \end{cases}$$

(4) if $\mathfrak{g} = \mathfrak{b}^\bullet$, then

$$\rho(q^{\pm E_a}) = \rho^+(q^{\pm E_a}) \otimes \rho^+(q^{\pm E_a}), \quad \rho(q^{2K}) = q \otimes q,$$

$$\rho(e_i) = \begin{cases} \psi_{\overline{m}}^* \otimes \rho(t_{\overline{m}}^{-1}) + 1 \otimes \psi_{\overline{m}}^*, & \text{if } i = \overline{m}, \\ \mathbf{e}_i^+ (\mathbf{t}_i^+)^{-1} + \mathbf{e}_i^+, & \text{if } i \neq \overline{m}, \end{cases}$$

$$\rho(f_i) = \begin{cases} (\psi_{\overline{m}} \otimes 1 + \rho(t_{\overline{m}}) \otimes \psi_{\overline{m}}) \sigma, & \text{if } i = \overline{m}, \\ \mathbf{f}_i^+ + \mathbf{t}_i^+ \mathbf{f}_i^+, & \text{if } i \neq \overline{m}. \end{cases}$$

Here ρ is as in (2) and σ is a $\mathbb{Q}(q)$ -linear operator on $\mathcal{F}_{q^2}^+ \otimes \mathcal{F}_{q^2}^+$ defined by

$$\sigma(\psi_{\mathbf{m}}|0\rangle \otimes \psi_{\mathbf{m}'}|0\rangle) = (-1)^{m_{\overline{m}} + m'_{\overline{m}}} \psi_{\mathbf{m}}|0\rangle \otimes \psi_{\mathbf{m}'}|0\rangle,$$

for $\mathbf{m} = (m_a)$, $\mathbf{m}' = (m'_a) \in \mathbf{B}^+$.

Proof. The proof is almost the same as in Lemma 5.2. So, we leave the details to the readers. \square

Corollary 5.4. We have $\mathcal{F}_q \cong \mathcal{F}_q^- \otimes \mathcal{F}_q^+$ as a module over $U_q(\mathfrak{gl}_{m|n}) \subset U_q(\mathfrak{c}_{m|n})$.

Proof. It follows immediately from comparing the actions of $U_q(\mathfrak{gl}_{m|n})$ in Lemma 5.2 and Proposition 5.3. \square

Corollary 5.5. $\mathcal{V}_q^{\otimes \ell}$ is completely reducible for $\ell \geq 1$.

Proof. By Proposition 5.3, $\mathcal{V}_q \in \mathcal{O}_q^{\text{int}}(m|n)$ with $\text{wt}(\mathcal{V}_q) \subset \ell\Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|n}} \mathbb{Z}_{\geq 0} \delta_a$ with $\ell = 1, 2$. By Theorem 4.2, $\mathcal{V}_q^{\otimes \ell} \in \mathcal{O}_q^{\text{int}}(m|n)$ and hence is completely reducible. \square

Put

$$\begin{aligned}\mathcal{L} &= \sum_{\mathbf{m} \in \mathbf{B}} \mathbb{A} \psi_{\mathbf{m}} |0\rangle, & \mathcal{B} &= \{ \pm \psi_{\mathbf{m}} |0\rangle \pmod{q\mathcal{L}} \mid \mathbf{m} \in \mathbf{B} \}, \\ \mathcal{L}^{\pm} &= \sum_{\mathbf{m} \in \mathbf{B}^{\pm}} \mathbb{A} \psi_{\mathbf{m}} |0\rangle, & \mathcal{B}^{\pm} &= \{ \pm \psi_{\mathbf{m}} |0\rangle \pmod{q\mathcal{L}} \mid \mathbf{m} \in \mathbf{B}^{\pm} \}.\end{aligned}$$

Theorem 5.6. *The following is a crystal base of \mathcal{V}_q .*

$$\begin{cases} (\mathcal{L}, \mathcal{B}), & \text{if } \mathfrak{g} = \mathfrak{c}, \\ (\mathcal{L}^+, \mathcal{B}^+), & \text{if } \mathfrak{g} = \mathfrak{b}, \mathfrak{d}, \\ (\mathcal{L}^+ \otimes \mathcal{L}^+, \mathcal{B}^+ \otimes \mathcal{B}^+), & \text{if } \mathfrak{g} = \mathfrak{b}^{\bullet}. \end{cases}$$

Proof. Suppose that $\mathfrak{g} = \mathfrak{c}$. It is clear by definition that \mathcal{L} is an \mathbb{A} -lattice of \mathcal{F}_q . We first claim that $(\mathcal{L}^{\pm}, \mathcal{B}^{\pm})$ is a crystal base of \mathcal{F}_q^{\pm} as a $U_q(\mathfrak{gl}_{m|n})$ -module, and hence $(\mathcal{L}, \mathcal{B})$ is a crystal base of \mathcal{F}_q as a $U_q(\mathfrak{gl}_{m|n})$ -module by Corollary 5.4. Consider $(\mathcal{L}^+, \mathcal{B}^+)$. The proof for $(\mathcal{L}^-, \mathcal{B}^-)$ is the same.

Let $a \in \mathbb{J}_{m|n}$ with $|a| = 1$ be given. Then for $m \geq 1$, $\psi_a^{(m)}$ is a highest weight vector with respect to $\langle e_a, f_a, t_a^{\pm 1} \rangle \cong U_q(\mathfrak{sl}_2)$. We have

$$\begin{aligned}\tilde{f}_a \psi_a^{(m)} |0\rangle &= f_a \psi_a^{(m)} |0\rangle = \mathbf{f}_a^+ \psi_a^{(m)} |0\rangle = \psi_{a+1} \psi_a^* \psi_a^{(m)} |0\rangle = \frac{1}{[m]} \psi_{a+1} [\omega_a] \psi_a^{(m-1)} |0\rangle \\ &= \frac{1}{[m]} \psi_{a+1} \psi_a^{(m-1)} [q^{-m+1} \omega_a] |0\rangle = \psi_a^{(m-1)} \psi_{a+1} |0\rangle.\end{aligned}$$

Similarly, we have

$$\tilde{f}_a^k \psi_a^{(m)} |0\rangle = f_a^{(k)} \psi_a^{(m)} |0\rangle = \psi_a^{(m-k)} \psi_{a+1}^{(k)} |0\rangle,$$

for $1 \leq k \leq m$. This implies that \mathcal{L}^+ is invariant under \tilde{e}_a and \tilde{f}_a for $a \in I_{0|n}$.

On the other hand, \mathcal{L}^+ is invariant under \tilde{e}_a and \tilde{f}_a for $a \in I_{m|0}$ since the weight of $\psi_{\mathbf{m}} |0\rangle$ with respect to $\langle e_a, f_a, t_a^{\pm 1} \rangle \cong U_q(\mathfrak{sl}_2)$ is minuscule for $\mathbf{m} \in \mathbf{B}^+$ and $a = \overline{m-1}, \dots, \overline{1}$. Also, we observe that for $m \geq 1$

$$e_0 \psi_{\frac{1}{2}}^{(m)} |0\rangle = \mathbf{e}_0^+ \psi_{\frac{1}{2}}^{(m)} |0\rangle = \psi_{\overline{1}} \psi_{\frac{1}{2}}^* \psi_{\frac{1}{2}}^{(m)} |0\rangle = \psi_{\overline{1}} \psi_{\frac{1}{2}}^{(m-1)} |0\rangle,$$

which implies that \mathcal{L}^+ is invariant under \tilde{e}_0 and \tilde{f}_0 . Therefore, \mathcal{L}^+ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n} \setminus \{\overline{m}\}$. It is straightforward to check the other conditions for $(\mathcal{L}^+, \mathcal{B}^+)$ to be a crystal base of \mathcal{F}_q^+ as a $U_q(\mathfrak{gl}_{m|n})$ -module. This proves our claim.

By Corollary 5.4, $(\mathcal{L}, \mathcal{B})$ is isomorphic to $(\mathcal{L}^- \otimes \mathcal{L}^+, \mathcal{B}^- \otimes \mathcal{B}^+)$ as a crystal base of the $U_q(\mathfrak{gl}_{m|n})$ -module \mathcal{F}_q , where $\psi_{\mathbf{m}^-} |0\rangle \otimes \psi_{\mathbf{m}^+} |0\rangle$ is mapped to $\psi_{\mathbf{m}^-} \psi_{\mathbf{m}^+} |0\rangle$ for $\mathbf{m}^{\pm} \in \mathbf{B}^{\pm}$.

Finally, for $\mathbf{m} \in \mathbf{B}$, the weight of $\psi_{\mathbf{m}}|0\rangle$ with respect to $\langle e_{\overline{m}}, f_{\overline{m}}, t_{\overline{m}}^{\pm 1} \rangle \cong U_q(\mathfrak{sl}_2)$ is minuscule, and hence \mathcal{L} is invariant under $\tilde{e}_{\overline{m}}$ and $\tilde{f}_{\overline{m}}$. For example, for \mathbf{m} with $m_a = 0$ when $a = \pm \overline{m}$

$$\tilde{f}_{\overline{m}}\psi_{\mathbf{m}}|0\rangle = f_{\overline{m}}\psi_{\mathbf{m}}|0\rangle = \pm\psi_{\mathbf{m}'}|0\rangle,$$

where $\mathbf{m}' = (m'_a) \in \mathbf{B}$ is such that $m'_a = 1$ for $a = \pm \overline{m}$ and $m'_a = m_a$ for $a \neq \pm \overline{m}$. Therefore, we conclude that $(\mathcal{L}, \mathcal{B})$ is a crystal base of \mathcal{F}_q .

We omit the proof for $\mathfrak{g} = \mathfrak{b}, \mathfrak{d}$ since it is similar to the case of $\mathfrak{g} = \mathfrak{c}$. Since the action of $U_q(\mathfrak{b}_{m|n})$ on $\mathcal{F}_{q^2}^+ \otimes \mathcal{F}_{q^2}^+$ is the same as $U_q(\mathfrak{b}_{m|n}^\bullet)$ except $f_{\overline{m}}$ by scalar multiplication, $(\mathcal{L}^+ \otimes \mathcal{L}^+, \mathcal{B}^+ \otimes \mathcal{B}^+)$ is also a crystal base of \mathcal{V}_q when $\mathfrak{g} = \mathfrak{b}^\bullet$. \square

Corollary 5.7. *For $\ell \geq 1$, $\mathcal{V}_q^{\otimes \ell}$ has a crystal base.*

6. ORTHOSYMPLECTIC TABLEAUX OF TYPE B AND C

In this section, we introduce our main combinatorial object called orthosymplectic tableaux, which play a crucial role in the next sections.

6.1. Semistandard tableaux. We assume that \mathcal{A} is a linearly ordered set with a \mathbb{Z}_2 -grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. When $\mathcal{A} = \mathbb{N}$, we assume that $\mathbb{N}_0 = \mathbb{N}$ with a usual linear ordering. For a skew Young diagram λ/μ , a tableau T obtained by filling λ/μ with entries in \mathcal{A} is called \mathcal{A} -semistandard if (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the entries in \mathcal{A}_0 (resp. \mathcal{A}_1) are strictly increasing in each column (resp. row). We say that the shape of T is λ/μ , and write $\text{sh}(T) = \lambda/\mu$. The weight of T is the sequence $(m_a)_{a \in \mathcal{A}}$, where m_a is the number of occurrences of a in T . We denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of all \mathcal{A} -semistandard tableaux of shape λ/μ (cf. [13]).

Let $x_{\mathcal{A}} = \{x_a | a \in \mathcal{A}\}$ be the set of formal commuting variables indexed by \mathcal{A} . For $\lambda \in \mathcal{P}$, let $s_{\lambda}(x_{\mathcal{A}})$ be the super Schur function corresponding to λ , which is the weight generating function of $SST_{\mathcal{A}}(\lambda)$, that is, $s_{\lambda}(x_{\mathcal{A}}) = \sum_T x_{\mathcal{A}}^T$, where $x_{\mathcal{A}}^T = \prod_a x_a^{m_a}$ and $(m_a)_{a \in \mathcal{A}}$ is the weight of T .

For $T \in SST_{\mathcal{A}}(\lambda)$ and $a \in \mathcal{A}$, we denote by $a \rightarrow T$ the tableau obtained by applying the usual Schensted column insertion of a into T (cf. [3, 13]). For a finite word $w = w_1 \dots w_r$ with letters in \mathcal{A} , we define $(w \rightarrow T) = (w_r \rightarrow (\dots (w_1 \rightarrow T)))$. For an \mathcal{A} -semistandard tableau S of skew Young diagram shape, let $w(S)$ be the word obtained by reading the entries of S column by column from right to left, where in each column we read the entries from top to bottom. We define $(S \rightarrow T) = (w(S) \rightarrow T)$. We denote by $w^{\text{rev}}(S)$ the reverse word of $w(S)$.

6.2. Combinatorial R -matrix. For a single-column shaped tableau S , let $S(i)$ ($i \geq 1$) denote the i -th entry from the bottom, and $\text{ht}(S)$ the height of S .

For $k, l \in \mathbb{Z}_{>0}$ with $k \geq l$, let us describe a bijection [22, Example 5.9]

$$(6.1) \quad R : SST_{\mathcal{A}}(1^k) \times SST_{\mathcal{A}}(1^l) \longrightarrow SST_{\mathcal{A}}(1^l) \times SST_{\mathcal{A}}(1^k),$$

which preserves the Knuth equivalence (cf. [2, Section 4.4]). It coincides with a combinatorial R -matrix when $\mathcal{A} = \mathcal{A}_0$ [30].

Let $(S, T) \in SST_{\mathcal{A}}(1^k) \times SST_{\mathcal{A}}(1^l)$ be given. We choose the entries $S(i_1), \dots, S(i_l)$ in S inductively as follows: (1) If $T(1) \in \mathcal{A}_0$ (resp. \mathcal{A}_1), then let $S(i_1)$ be the largest one which is no greater than (resp. less than) $T(1)$. If there is no such entry, then put $i_1 = 1$. (2) For $2 \leq u \leq l$, if $T(u) \in \mathcal{A}_0$ (resp. \mathcal{A}_1), then let $S(i_u)$ be the largest one in $\{S(1), \dots, S(k)\} \setminus \{S(i_1), \dots, S(i_{u-1})\}$ which is no greater than (resp. less than) $T(u)$. If there is no such entry, then choose $S(i_u)$ to be the largest one in $\{S(1), \dots, S(k)\} \setminus \{S(i_1), \dots, S(i_{u-1})\}$ at the lowest position. Then we have

$$R(S, T) = (T^\sharp, S^\sharp),$$

where T^\sharp is the tableau of shape (1^l) having $\{S(i_1), \dots, S(i_l)\}$ as its entries after rearrangement with respect to the ordering on \mathcal{A} , and S^\sharp is the tableau of shape (1^k) obtained from S by replacing with $\{S(i_1), \dots, S(i_l)\}$ with $\{T(1), \dots, T(l)\}$ and rearranging the entries.

Remark 6.1. In particular, if the pair (S, T) forms an \mathcal{A} -semistandard tableau U of shape $(k, l)'$, then $(T^\sharp \rightarrow S^\sharp) = U$.

6.3. Signature of a two-column shaped tableau. Let us first recall a combinatorial algorithm often called signature rule. Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be a sequence (not necessarily of finite length) with $\sigma_i \in \{+, -, \cdot\}$ such that $\sigma_i = +$ or \cdot for $i \gg 0$. Then we replace a pair $(\sigma_j, \sigma_{j'}) = (+, -)$ in σ , where $j < j'$ and $\sigma_j'' = \cdot$ for $j < j'' < j'$, with (\cdot, \cdot) , and repeat this process as far as possible until we get a sequence with no $-$ placed to the right of $+$. We denote the resulting reduced sequence by $\tilde{\sigma}$.

Let $w = w_1 \dots w_r$ be a finite word with letters in \mathbb{N} . Fix $k \in \mathbb{N}$. We associate a sequence $\sigma = (\sigma_1, \dots, \sigma_r)$, where $\sigma_i = +$ (resp. $-$) if $w_i = k$ (resp. $k+1$) and $\sigma_i = \cdot$ otherwise, for $1 \leq i \leq r$. We say that the k -signature of w is (a, b) , where a (resp. b) is the number of $-$'s (resp. $+$'s) in $\tilde{\sigma}$. For an \mathbb{N} -semistandard tableau T , we define the k -signature of T to be that of $w(T)$.

Let S_1 and S_2 be single-column shaped \mathcal{A} -semistandard tableaux. Let $w = w(S_2)w(S_1)$ with $w(S_2) = w_1 \dots w_r$ and $w(S_1) = w_{r+1} \dots w_s$. Given $U \in SST_{\mathcal{A}}(\mu)$

$(\mu \in \mathcal{P})$, let

$$P(S_1, S_2; U) = (S_1 \rightarrow (S_2 \rightarrow U)).$$

Suppose that $\lambda = \text{sh}(P(S_1, S_2; U))$. Define

$$Q(S_1, S_2; U)_{[k]}$$

to be a tableau of shape λ'/μ' with entries in $\{k, k+1\}$ such that $\text{sh}(w_1 \dots w_i \rightarrow U)'/\text{sh}(w_1 \dots w_{i-1} \rightarrow U)'$ is filled with k (resp. $k+1$) for $1 \leq i \leq r$ (resp. $r+1 \leq i \leq s$). We have $Q(S_1, S_2; U)_{[k]} \in SST_{\{k, k+1\}}(\lambda'/\mu')$ by definition. The correspondence $(S_1, S_2) \mapsto (P(S_1, S_2; U), Q(S_1, S_2; U)_{[k]})$ is reversible and hence one-to-one by reverse bumping.

We define the *signature of* (S_1, S_2) to be the k -signature of $Q(S_1, S_2; U)_{[k]}$. Indeed, if (a, b) is the k -signature of $Q(S_1, S_2; U)_{[k]}$, then we can check by standard arguments (cf. [13, Chapter1]) that it does not depend on the choice of U . In particular, if $U = \emptyset$, then $\lambda = (r+a, r-b)'$ with $a + (r-b) = s$. This means that $r-b$ entries in S_2 are bumped out and $s - (r-b)$ entries in S_1 are placed below the bottom of S_2 when S_1 is inserted into S_2 .

For $a, b, c \in \mathbb{Z}_{\geq 0}$, let

$$\lambda(a, b, c) = (2^{b+c}, 1^a)/(1^b) = (a+b+c, b+c)'/(b)',$$

which is a skew Young diagram with two columns of heights $a+c$ and $b+c$. For example,

$$\lambda(1, 3, 2) = \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array}$$

Let S be a tableau of shape $\lambda(a, b, c)$, whose column is \mathcal{A} -semistandard. For $i = 1, 2$, let S_i be the i -th column of S from the left.

Lemma 6.2. Under the above hypothesis, S is \mathcal{A} -semistandard of shape $\lambda(a, b, c)$ if and only if the signature of (S_1, S_2) is $(a-p, b-p)$ for some $0 \leq p \leq \min\{a, b\}$.

Proof. Suppose that S is \mathcal{A} -semistandard. Let $w_1 \dots w_{b+c}$ (resp. $w_{b+c+1} \dots w_{a+b+2c}$) be the subword of $w(S)$ corresponding to the entries in S_2 (resp. S_1). Given $U \in SST_{\mathcal{A}}(\mu)$, let i_s ($1 \leq s \leq b+c$) be the row index (enumerated from the top) of 1 in $Q(S_1, S_2; U)_{[1]}$ corresponding to w_s and j_t ($1 \leq t \leq a+c$) the row

index of 2 in $Q(S_1, S_2; U)_{[1]}$ corresponding to w_{t+b+c} . Note that $i_1 \geq \dots \geq i_{b+c}$ and $j_1 \geq \dots \geq j_{a+c}$.

Since S is \mathcal{A} -semistandard, we have $i_{b+u} < j_u$ for $1 \leq u \leq c$, which implies that the cancellation of $(+, -)$ or $(1, 2)$ -pair occurs in $w = w(Q(S_1, S_2; U)_{[1]})$ at least as many as c times. Hence the 1-signature of w is $(a - p, b - p)$ for some $0 \leq p \leq \min\{a, b\}$.

Conversely, suppose that S is not \mathcal{A} -semistandard. We can choose $p \geq 1$ such that there exists $\tilde{S} \in SST_{\mathcal{A}}(\lambda(a + p, b + p, c - p))$ whose columns are the same as those of S . We assume that p is minimal among such ones. Then by the minimality of p , we can check that $j_1 \leq i_{b+p}$ and hence the 1-signature of w is $(a + p, b + p)$, which is a contradiction. \square

6.4. Combinatorial R -matrix and recording tableaux. Let us define an operator r_k on a finite word w with letters in \mathbb{N} . Let w be given with the k -signature (a, b) , and let σ and $\tilde{\sigma}$ denote the associated sequences of $+, -, \cdot$'s defined in Section 6.3.

If $a \leq b$, then we define $r_k w$ to be the word obtained by replacing k corresponding to the leftmost $(b - a) +$'s in $\tilde{\sigma}$ with $k + 1$. If $a \geq b$ then we define $r_k w$ to be the word obtained by replacing $k + 1$ corresponding to the rightmost $(a - b) -$'s in $\tilde{\sigma}$ with k . Then the k -signature of $r_k w$ is (b, a) in both cases. Note that r_k is the Weyl group action on the tensor product of the crystal of the natural representation of $U_q(\mathfrak{sl}_2)$ ($k \xrightarrow{k} k + 1$) with respect to the tensor product rule (5.1).

Also, we define $\varrho_k w$ to be the word obtained by replacing $k + 1$'s in w , which do not come from a cancelled pair $(+, -)$ in σ , with k 's. Note that the k -signature of $\varrho_k w$ is $(0, a + b)$, and $\varrho_k w = r_k w$ when $b = 0$.

For an \mathbb{N} -semistandard tableau T , we define $r_k(T)$ (resp. $\varrho_k(T)$) to be the tableau obtained from T by applying r_k (resp. ϱ_k) to $w(T)$, that is, $w(r_k(T)) = r_k(w(T))$ and $w(\varrho_k(T)) = \varrho_k(w(T))$. Note that $r_k(T)$ and $\varrho_k(T)$ are well defined \mathbb{N} -semistandard tableaux.

Now, let $S \in SST_{\mathcal{A}}(\lambda(a, b, c))$ be given, and let S_i be the i -th column of S from the left ($i = 1, 2$). Suppose first that $b = 0$. Let

$$(T_1, T_2) = R(S_1, S_2) \in SST_{\mathcal{A}}(1^c) \times SST_{\mathcal{A}}(1^{a+c}),$$

where R is the combinatorial R -matrix in (6.1). The signature of (S_1, S_2) is $(a, 0)$ by Lemma 6.2, and the signature of (T_1, T_2) is $(0, a)$. Let $U \in SST_{\mathcal{A}}(\lambda)$ be given with $\lambda \in \mathcal{P}$. By considering the bumping paths in the insertion of $(S_1 \rightarrow (S_2 \rightarrow U))$ and the definition of R , we have the following.

Lemma 6.3. *Under the above hypothesis,*

$$r_k Q(S_1, S_2; U)_{[k]} = Q(T_1, T_2; U)_{[k]}.$$

Next, we suppose that $b \geq 0$ and the signature of (S_1, S_2) is (a, b) . Let $\tilde{S} = (S_1 \rightarrow S_2) \in SST_{\mathcal{A}}(\lambda(a+b, 0, c))$ with \tilde{S}_i the i -th column of \tilde{S} from the left ($i = 1, 2$). Let

$$(T_1, T_2) = R(\tilde{S}_1, \tilde{S}_2) \in SST_{\mathcal{A}}(1^c) \times SST_{\mathcal{A}}(1^{a+b+c}).$$

Lemma 6.4. *Under the above hypothesis,*

$$\varrho_k Q(S_1, S_2; U)_{[k]} = Q(T_1, T_2; U)_{[k]}.$$

Proof. Let S_2^{low} be the subtableau of S_2 consisting of $S_2(i)$ for $1 \leq i \leq c$, and let S_2^{up} be its complement in S_2 . Then (S_1, S_2^{low}) forms an \mathcal{A} -semistandard tableau of shape $\lambda(a, 0, c)$. Let $(V_1, V_2^{\text{low}}) = R(S_1, S_2^{\text{low}}) \in SST_{\mathcal{A}}(1^c) \times SST_{\mathcal{A}}(1^{a+c})$.

Put $U^* = (S_2^{\text{up}} \rightarrow U)$ and $\mu = \text{sh}(U^*)$. Let Q_1 be the tableau of shape μ'/λ' filled with k and let $Q_2 = Q(S_1, S_2^{\text{low}}; U^*)_{[k]}$, which is of shape ν'/μ' with $\nu = \text{sh}(S_1 \rightarrow (S_2^{\text{low}} \rightarrow U^*))$. We see by definition that the tableau Q obtained by glueing Q_1 and Q_2 is equal to $Q(S_1, S_2; U)_{[k]}$.

By Lemma 6.3, we have $r_k Q(S_1, S_2^{\text{low}}; U^*)_{[k]} = Q(V_1, V_2^{\text{low}}; U^*)_{[k]}$. Also we see that each $k+1$ in Q_2 , which is replaced by k under r_k , is always to the east of k 's in Q_1 , since the k -signature of Q is (a, b) . This implies that the tableau obtained by glueing Q_1 and $r_k Q_2$ is equal to $\varrho_k Q$. Hence, if V_2 is the single column tableau of height $a+b+c$ obtained by placing V_2^{low} below S_2^{up} , then V_2 is \mathcal{A} -semistandard, and $Q(V_1, V_2; U)_{[k]} = \varrho_k Q$.

Since $(V_1 \rightarrow V_2)$ and \tilde{S} are Knuth equivalent, we have $(V_1 \rightarrow V_2) = \tilde{S}$. Furthermore, $V_1 = T_1$ and $V_2 = T_2$, since $(T_1 \rightarrow T_2) = \tilde{S}$ and there exist unique $(Y_1, Y_2) \in SST_{\mathcal{A}}(1^c) \times SST_{\mathcal{A}}(1^{a+b+c})$ such that $(Y_1 \rightarrow Y_2) = \tilde{S} \in SST_{\mathcal{A}}(\lambda(a+b, 0, c))$. Therefore, we have $\varrho_k Q(S_1, S_2; U)_{[k]} = Q(V_1, V_2; U)_{[k]} = Q(T_1, T_2; U)_{[k]}$. \square

6.5. Orthosymplectic tableaux of type B and C . Let us assume that $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$. From now on, for a two-column shaped \mathcal{A} -semistandard tableau T , we denote by T^{L} and T^{R} the left and right columns of T , respectively, and often identify T with $(T^{\text{L}}, T^{\text{R}})$.

Definition 6.5. For $a \in \mathbb{Z}_{\geq 0}$, we define $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(a)$ to be the set of \mathcal{A} -semistandard tableaux T of shape $\lambda(a, b, c)$ such that

- (1) $(b, c) \in \{0\} \times \mathbb{Z}_{\geq 0}$ if $\mathfrak{g} = \mathfrak{c}$,
- (2) $(b, c) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and the signature of $(T^{\text{L}}, T^{\text{R}})$ is (a, b) if $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$.

We define $\mathbf{T}_{\mathcal{A}}^{\text{sp}}$ to be the set of \mathcal{A} -semistandard tableaux of shape (1^a) for some $a \in \mathbb{Z}_{\geq 0}$.

Let $T \in \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(a)$ be given with $\text{sh}(T) = \lambda(a, b, c)$ for some $a, b, c \geq 0$. Let us define

$$({}^{\text{L}}T, {}^{\text{R}}T) = R(T^{\text{L}} \rightarrow T^{\text{R}}).$$

Since $\text{sh}(T^{\text{L}} \rightarrow T^{\text{R}}) = \lambda(a + b, 0, c)$ by Definition 6.5, which is of two column shape, $R(T^{\text{L}} \rightarrow T^{\text{R}})$ is well defined and $({}^{\text{L}}T, {}^{\text{R}}T) \in SST_{\mathcal{A}}(1^c) \times SST_{\mathcal{A}}(1^{a+b+c})$. Note that for $\mathfrak{g} = \mathfrak{c}$, we have $({}^{\text{L}}T, {}^{\text{R}}T) = R(T^{\text{L}}, T^{\text{R}})$ since $(T^{\text{L}} \rightarrow T^{\text{R}}) = T$. For $T \in \mathbf{T}_{\mathcal{A}}^{\text{sp}}$, we identify T with $(\emptyset, T) = (T^{\text{L}}, T^{\text{R}})$, and hence ${}^{\text{L}}T = \emptyset, {}^{\text{R}}T = T$.

Example 6.6. Suppose that $\mathcal{A} = \mathbb{J}_{4|\infty}$.

$$\begin{aligned} \mathbf{T}_{\mathcal{A}}^{\mathfrak{b}}(2) \ni S = (S^{\text{L}}, S^{\text{R}}) &= \begin{array}{|c|c|} \hline \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{2} \\ \hline \overline{1} & \frac{3}{2} \\ \hline \frac{1}{2} & \\ \hline \frac{1}{2} & \\ \hline \end{array} & ({}^{\text{L}}S, {}^{\text{R}}S) = \begin{array}{|c|c|} \hline \overline{3} & \overline{4} \\ \hline \overline{1} & \overline{3} \\ \hline \frac{1}{2} & \overline{2} \\ \hline & \frac{1}{2} \\ \hline & \frac{3}{2} \\ \hline \end{array} \\ \\ \mathbf{T}_{\mathcal{A}}^{\mathfrak{b}}(3) \ni T = (T^{\text{L}}, T^{\text{R}}) &= \begin{array}{|c|c|} \hline & \overline{4} \\ \hline & \overline{3} \\ \hline \overline{3} & \overline{2} \\ \hline \overline{1} & \overline{1} \\ \hline \frac{1}{2} & \frac{5}{2} \\ \hline \frac{3}{2} & \\ \hline \frac{3}{2} & \\ \hline \frac{5}{2} & \\ \hline \end{array} & (T^{\text{L}} \rightarrow T^{\text{R}}) = \begin{array}{|c|c|} \hline \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{1} \\ \hline \overline{2} & \frac{5}{2} \\ \hline \overline{1} & \\ \hline \frac{1}{2} & \\ \hline \frac{3}{2} & \\ \hline \frac{3}{2} & \\ \hline \frac{5}{2} & \\ \hline \end{array} & ({}^{\text{L}}T, {}^{\text{R}}T) = \begin{array}{|c|c|} \hline & \overline{4} \\ \hline & \overline{3} \\ \hline \overline{3} & \overline{2} \\ \hline \overline{1} & \overline{1} \\ \hline \frac{3}{2} & \frac{1}{2} \\ \hline & \frac{3}{2} \\ \hline & \frac{5}{2} \\ \hline & \frac{5}{2} \\ \hline \end{array} \end{aligned}$$

Definition 6.7. Suppose that $S \in \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(p)$ or $\mathbf{T}_{\mathcal{A}}^{\text{sp}}$, and $T \in \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(q)$ are given for $p \leq q$. We assume that $p = 0$ when $T \in \mathbf{T}_{\mathcal{A}}^{\text{sp}}$. We say that the pair (S, T) is *admissible* and write $S \prec T$ if it satisfies the following conditions:

- (1) $\text{ht}(S^{\text{R}}) + p \leq \text{ht}(T^{\text{L}})$,
- (2) ${}^{\text{R}}S(i) \leq T^{\text{L}}(i)$ for $i \geq 1$,
- (3) $S^{\text{R}}(i + q - p) \leq {}^{\text{L}}T(i)$ for $i \geq 1$,

where the equality holds in (2) and (3) only if the entries are even or in \mathcal{A}_0 .

Example 6.8. Let S and T be as in Example 6.6. Then

$$({}^{\text{R}}S, T^{\text{L}}) = \begin{array}{|c|c|} \hline & \overline{3} \\ \hline \overline{4} & \overline{1} \\ \hline \overline{3} & \frac{1}{2} \\ \hline \overline{2} & \frac{3}{2} \\ \hline \frac{1}{2} & \frac{3}{2} \\ \hline \frac{3}{2} & \frac{5}{2} \\ \hline \end{array} \quad (S^{\text{R}}, {}^{\text{L}}T) = \begin{array}{|c|c|} \hline & \overline{3} \\ \hline \overline{3} & \overline{1} \\ \hline \overline{2} & \frac{3}{2} \\ \hline \frac{3}{2} & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array}$$

Hence (S, T) is admissible, or $S \prec T$.

Remark 6.9. The conditions (2) and (3) in Definition 6.7 are equivalent to saying that $({}^R S, T^L)$ and $(S^R, {}^L T)$ form \mathcal{A} -semistandard tableaux of shape $\lambda(a, b, c)$ and $\lambda(a^*, b^*, c^*)$, respectively, where

$$\begin{cases} a = 0, \\ b = \text{ht}(T^L) - \text{ht}(S^L), \\ c = \text{ht}(S^L), \end{cases} \quad \begin{cases} a^* = q - p, \\ b^* = \text{ht}(T^L) - \text{ht}(S^R) - p, \\ c^* = \text{ht}(S^R) + p - q. \end{cases}$$

Now we introduce the notion of orthosymplectic tableaux, which is our main combinatorial object.

Definition 6.10. Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$ be given. Let

$$(6.2) \quad L = \begin{cases} \ell, & \text{if } \mathfrak{g} = \mathfrak{c}, \\ \ell/2, & \text{if } \mathfrak{g} = \mathfrak{b}^\bullet \text{ or } \mathfrak{g} = \mathfrak{b} \text{ with } \ell - 2\lambda_1 \text{ even,} \\ (\ell + 1)/2, & \text{if } \mathfrak{g} = \mathfrak{b} \text{ with } \ell - 2\lambda_1 \text{ odd.} \end{cases}$$

We define $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, \ell)$ to be the set of $\mathbf{T} = (T_L, \dots, T_1)$ in

$$\begin{cases} \mathbf{T}_{\mathcal{A}}^{\text{sp}} \times \mathbf{T}_{\mathcal{A}}^{\mathfrak{b}}(\lambda'_{L-1}) \times \dots \times \mathbf{T}_{\mathcal{A}}^{\mathfrak{b}}(\lambda'_1), & \text{if } \mathfrak{g} = \mathfrak{b} \text{ with } \ell - 2\lambda_1 \text{ odd,} \\ \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda'_L) \times \dots \times \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda'_1), & \text{if otherwise,} \end{cases}$$

such that $T_{k+1} \prec T_k$ for $1 \leq k \leq L-1$. We call $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, \ell)$ an *orthosymplectic tableau of type \mathfrak{g} and shape (λ, ℓ)* .

Note that $\mathcal{P}(\mathfrak{b}^\bullet) \subsetneq \mathcal{P}(\mathfrak{b})$ and $\mathbf{T}_{\mathcal{A}}^{\mathfrak{b}^\bullet}(\lambda, \ell) = \mathbf{T}_{\mathcal{A}}^{\mathfrak{b}}(\lambda, \ell)$ as a set for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{b}^\bullet)$.

Let z be an indeterminate.

6.6. Schur positivity. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$, put

$$S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathcal{A}}) = z^{\ell} \sum_{\mathbf{T} \in \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, \ell)} \prod_{k=1}^L x_{\mathcal{A}}^{T_k},$$

which is the weight generating function of $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, \ell)$. We will show that $S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathcal{A}})$ can be written as a non-negative integral (possibly infinite) linear combination of $s_{\mu}(x_{\mathcal{A}})$. For this, we introduce the following.

Definition 6.11. Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$ be given with L as in (6.2). For $\mu \in \mathcal{P}$, we define $\mathbf{K}_{\mu(\lambda, \ell)}^{\mathfrak{g}}$ to be the set of $Q \in SST_{\{1, \dots, 2L\}}(\mu)$ with weight (m_1, \dots, m_{2L}) satisfying the following conditions:

- $\mathfrak{g} = \mathfrak{c}$

- (1) $m_{2k} - m_{2k-1} = \lambda'_k$ for $1 \leq k \leq L$,
- (2) $m_{2k} \geq m_{2k+2}$ for $1 \leq k \leq L-1$,
- (3) the $(2k-1)$ -signature of Q is $(\lambda'_k, 0)$ for $1 \leq k \leq L$,
- (4) the $2k$ -signature of $r_{2k+1}Q$ is $(0, m_{2k} - m_{2k+2})$ for $1 \leq k \leq L-1$,
- (5) the $2k$ -signature of $r_{2k-1}Q$ is $(\lambda'_k - \lambda'_{k+1} - p, m_{2k} - m_{2k+2} - p)$ with $p \geq 0$ for $1 \leq k \leq L-1$.

• $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$

- (1) $m_{2L} = 0$ if $\ell - 2\lambda_1$ is odd,
- (2) $m_{2k-1} \geq m_{2k} - \lambda'_k \geq 0$ for $1 \leq k \leq L$,
- (3) $m_{2k} \geq m_{2k+1} + \lambda'_{k+1}$ for $1 \leq k \leq L-1$,
- (4) the $(2k-1)$ -signature of Q is $(\lambda'_k, m_{2k-1} - m_{2k} + \lambda'_k)$ for $1 \leq k \leq L$,
- (5) the $2k$ -signature of $\varrho_{2k+1}Q$ is $(0, m_{2k} - m_{2k+1} - \lambda'_{k+1})$ for $1 \leq k \leq L-1$,
- (6) the $2k$ -signature of $\varrho_{2k-1}Q$ is $(\lambda'_k - \lambda'_{k+1} - p, m_{2k} - m_{2k+1} - \lambda'_{k+1} - p)$ with $p \geq 0$ for $1 \leq k \leq L-1$.

Then we have the following.

Theorem 6.12. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$, we have a weight preserving bijection*

$$\psi_{(\lambda, \ell)} : \mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, \ell) \longrightarrow \bigsqcup_{\mu \in \mathcal{P}} SST_{\mathcal{A}}(\mu) \times \mathbf{K}_{\mu(\lambda, \ell)}^{\mathfrak{g}}.$$

Proof. Let us first prove the case when $\mathfrak{g} = \mathfrak{c}$.

Let $\mathbf{T} = (T_\ell, \dots, T_1) \in \mathbf{T}_{\mathcal{A}}^{\mathfrak{c}}(\lambda'_\ell) \times \dots \times \mathbf{T}_{\mathcal{A}}^{\mathfrak{c}}(\lambda'_1)$ be given. Put $P_1 = T_1^{\mathbf{R}}$ and $P_2 = (T_1^{\mathbf{L}} \rightarrow P_1)$, and define inductively

$$P_{2k-1} = (T_k^{\mathbf{R}} \rightarrow P_{2k-2}), \quad P_{2k} = (T_k^{\mathbf{L}} \rightarrow P_{2k-1}),$$

for $2 \leq k \leq \ell$. Let $P = P_{2\ell}$. Suppose that $\mu = \text{sh}(P)'$. Define Q to be a tableau in $SST_{\{1, \dots, 2\ell\}}(\mu)$ such that $\text{sh}(P_k)'/\text{sh}(P_{k-1})'$ is filled with k for $1 \leq k \leq 2\ell$, where we assume that $P_0 = \emptyset$. Note that the subtableau of Q with entries $\{2k-1, 2k\}$ (resp. $\{2k, 2k+1\}$) is $Q(T_k^{\mathbf{L}}, T_k^{\mathbf{R}}; P_{2k-2})_{[2k-1]}$ (resp. $Q(T_{k+1}^{\mathbf{R}}, T_k^{\mathbf{L}}; P_{2k-1})_{[2k]}$) for $1 \leq k \leq \ell$ (resp. $1 \leq k \leq \ell-1$).

Let $(m_1, \dots, m_{2\ell})$ be the weight of Q . Then m_k 's satisfy the conditions (1) and (2) for $\mathbf{K}_{\mu(\lambda, \ell)}^{\mathfrak{c}}$ in Definition 6.11 since $m_{2k-1} = \text{ht}(T_k^{\mathbf{R}})$ and $m_{2k} = \text{ht}(T_k^{\mathbf{L}})$ for $1 \leq k \leq \ell$. Also, by Lemma 6.2, Q satisfies the condition (3) since each T_k is \mathcal{A} -semistandard of shape $\lambda(m_{2k} - m_{2k-1}, 0, m_{2k-1})$, where $m_{2k} - m_{2k-1} = \lambda'_k$.

For $1 \leq k \leq \ell$, let us define a sequence of tableaux $P_i^{(k)}$ ($1 \leq i \leq 2\ell$) in the same way as P_i 's except

$$P_{2k-1}^{(k)} = ({}^{\mathbf{R}}T_k \rightarrow P_{2k-2}^{(k)}), \quad P_{2k}^{(k)} = ({}^{\mathbf{L}}T_k \rightarrow P_{2k-1}^{(k)}).$$

Since $(T_k^L \rightarrow T_k^R) = ({}^L T_k \rightarrow {}^R T_k) = T_k$, we have $P_i^{(k)} = P_i$ for $i \neq 2k-1, 2k$. We define $Q^{(k)}$ in a similar way using $P_i^{(k)}$'s. By Lemma 6.3, we have $Q^{(k)} = r_{2k-1}Q$. Then by Lemma 6.2, the conditions (2) and (3) in Definition 6.7 imply the conditions (4) and (5) in Definition 6.11, respectively. Hence, we have $Q \in \mathbf{K}_{\mu(\lambda, \ell)}^c$.

Now, we define $\psi_{(\lambda, \ell)}(\mathbf{T}) = (P, Q)$. Since the construction of (P, Q) is reversible, it is not difficult to see that $\psi_{(\lambda, \ell)}$ is a bijection to the set of pairs (P, Q) with $P \in SST_{\mathcal{A}}(\mu)$ and $Q \in \mathbf{K}_{\mu(\lambda, \ell)}^c$ for $\mu \in \mathcal{P}$.

The proof for the case when $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$ is almost identical, where we replace r_k with ϱ_k and use Lemma 6.4 instead of Lemma 6.3. We leave the details to the reader. \square

Corollary 6.13. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$, we have*

$$S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathcal{A}}) = z^\ell \sum_{\mu \in \mathcal{P}} K_{\mu(\lambda, \ell)}^{\mathfrak{g}} s_\mu(x_{\mathcal{A}}),$$

where $K_{\mu(\lambda, \ell)}^{\mathfrak{g}}$ is the number of tableaux in $\mathbf{K}_{\mu(\lambda, \ell)}^{\mathfrak{g}}$.

7. CHARACTER OF A HIGHEST WEIGHT MODULE IN $\mathcal{O}_q^{int}(m|n)$

In this section, we show that the weight generating function of $\mathbf{T}_{m|n}(\lambda, \ell)$ is equal to the character of $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$, where $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$.

7.1. Combinatorial formula of irreducible characters in $\mathcal{O}_q^{int}(m|n)$. We assume that $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$. Let $P_{m+n} = \bigoplus_{a \in \mathbb{J}_{m+n}} \mathbb{Z}\delta_a \oplus \mathbb{Z}\Lambda_{\overline{m}}$ and $P_{m+n}^+ = P_{m+\infty}^+ \cap P_{m+n}$. Let $\mathcal{P}(\mathfrak{g})_{m+n}$ be the set of $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$ such that $\Lambda_{m+\infty}(\lambda, \ell) \in P_{m+n}^+$. Let us write $\Lambda_{m+n}(\lambda, \ell) = \Lambda_{m+\infty}(\lambda, \ell)$ for $\Lambda_{m+\infty}(\lambda, \ell) \in P_{m+n}^+$. Let $U_q(\mathfrak{g}_{m+n})$ be the quantized enveloping algebra associated to \mathfrak{g}_{m+n} and $L_q(\mathfrak{g}_{m+n}, \Lambda)$ its irreducible highest weight module with highest weight $\Lambda \in P_{m+n}$.

For simplicity, we put

$$\mathbf{T}_{m+n}(a) = \mathbf{T}_{\mathbb{J}_{m+n}}^{\mathfrak{g}}(a), \quad \mathbf{T}_{m+n}(\lambda, \ell) = \mathbf{T}_{\mathbb{J}_{m+n}}^{\mathfrak{g}}(\lambda, \ell), \quad \mathbf{T}_{m+n}^{\text{sp}} = \mathbf{T}_{\mathbb{J}_{m+n}}^{\text{sp}},$$

for $a \in \mathbb{Z}_{\geq 0}$ and $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m+n}$. We will first define an (abstract) \mathfrak{g}_{m+n} -crystal structure on $\mathbf{T}_{m+n}(\lambda, \ell)$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m+n}$, and show that it is isomorphic to the crystal of $L_q(\mathfrak{g}_{m+n}, \Lambda_{m+n}(\lambda, \ell))$.

We refer the reader to [12, 24] and references therein for general exposition on (abstract) crystals associated to a symmetrizable Kac-Moody algebra. Note that $\mathfrak{b}_{m+n}^\bullet$ is a Kac-Moody superalgebra, and one can consider abstract $\mathfrak{b}_{m+n}^\bullet$ -crystals in the same way (cf. [17]). To avoid confusion with those for $U_q(\mathfrak{g}_{m|n})$, we denote the Kashiwara operators on \mathfrak{g}_{m+n} -crystals by \tilde{e}_i and \tilde{f}_i for $i \in I_{m+n}$, and assume that the tensor product rule in this case follows (5.1).

Recall that \mathbb{J}_{m+n} can be identified with the crystal of the natural representation of $U_q(\mathfrak{gl}_{m+n})$ with respect to \tilde{e}_i and \tilde{f}_i for $i \in I_{m+n} \setminus \{\overline{m}\}$, where

$$\begin{aligned} \overline{m} &\xrightarrow{\overline{m-1}} \overline{m-1} \xrightarrow{\overline{m-2}} \cdots \xrightarrow{\overline{1}} \overline{1} \xrightarrow{0} 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots & (n = \infty), \\ \overline{m} &\xrightarrow{\overline{m-1}} \overline{m-1} \xrightarrow{\overline{m-2}} \cdots \xrightarrow{\overline{1}} \overline{1} \xrightarrow{0} 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n & (n < \infty), \end{aligned}$$

where $\text{wt}(a) = \delta_a$ for $a \in \mathbb{J}_{m+n}$. Then the set of finite words with letters in \mathbb{J}_{m+n} has a \mathfrak{gl}_{m+n} -crystal structure, where each non-empty word $w = w_1 \cdots w_r$ is identified with $w_1 \otimes \cdots \otimes w_r \in (\mathbb{J}_{m+n})^{\otimes r}$. Also, by applying \tilde{e}_i and \tilde{f}_i to the word of an \mathbb{J}_{m+n} -semistandard tableau, we have an (abstract) \mathfrak{gl}_{m+n} -crystal structure on $SST_{\mathbb{J}_{m+n}}(\lambda/\mu)$ for a skew Young diagram λ/μ [26]. For $\lambda \in \mathcal{P}$, we denote by H_λ the highest weight element in $SST_{\mathbb{J}_{m+n}}(\lambda)$.

Let \mathcal{B} denote either $\mathbf{T}_{m+n}^{\text{sp}}$ or $\mathbf{T}_{m+n}(a)$ for $0 \leq a \leq m+n$. Let us define a \mathfrak{g}_{m+n} -crystal structure on \mathcal{B} .

For $T \in \mathcal{B}$, let

$$\text{wt}(T) = \begin{cases} r\Lambda_{\overline{m}} + \sum_{s \in \mathbb{J}_{m+n}} m_s \delta_s, & \text{if } \mathcal{B} = \mathbf{T}_{m+n}(a) \text{ for } 0 \leq a \leq m+n, \\ \Lambda_{\overline{m}} + \sum_{s \in \mathbb{J}_{m+n}} m_s \delta_s, & \text{if } \mathcal{B} = \mathbf{T}_{m+n}^{\text{sp}}, \end{cases}$$

where $r = 1$ for $\mathfrak{g} = \mathfrak{c}$ and $r = 2$ otherwise, and $(m_s)_{s \in \mathbb{J}_{m+n}}$ is the content of T . Since \mathcal{B} is a set of \mathbb{J}_{m+n} -semistandard tableaux, it is a \mathfrak{gl}_{m+n} -crystal with respect to \tilde{e}_i and \tilde{f}_i for $i \in I_{m+n} \setminus \{\overline{m}\}$, where ε_i and φ_i are defined in a usual way. So it suffices to define $\tilde{e}_{\overline{m}}$ and $\tilde{f}_{\overline{m}}$.

CASE 1. Suppose that $\mathfrak{g} = \mathfrak{c}$ and $\mathcal{B} = \mathbf{T}_{m+n}(a)$ for $0 \leq a \leq m+n$. For $T \in \mathbf{T}_{m+n}(a)$, let t^L and t^R be the top entries in T^L and T^R , respectively. If $t^L = t^R = \overline{m}$, then we define $\tilde{e}_{\overline{m}}T$ to be the tableau obtained by removing $\boxed{\overline{m}}\boxed{\overline{m}}$ from T . Otherwise, we define $\tilde{e}_{\overline{m}}T = \mathbf{0}$. Also, if $t^L, t^R > \overline{m}$, then we define $\tilde{f}_{\overline{m}}T$ to be the tableau obtained by adding $\boxed{\overline{m}}\boxed{\overline{m}}$ on top of T . Otherwise, we define $\tilde{f}_{\overline{m}}T = \mathbf{0}$. Here $\mathbf{0}$ denotes a formal symbol conventionally used in abstract crystals.

CASE 2. Suppose that $\mathfrak{g} = \mathfrak{b}$ and $\mathcal{B} = \mathbf{T}_{m+n}^{\text{sp}}$. For $T \in \mathbf{T}_{m+n}^{\text{sp}}$, let t be the top entry of T . If $t = \overline{m}$, then we define $\tilde{e}_{\overline{m}}T$ to be the tableau obtained by removing $\boxed{\overline{m}}$ from T . Otherwise, we define $\tilde{e}_{\overline{m}}T = \mathbf{0}$. We define $\tilde{f}_{\overline{m}}T$ in a similar way by adding $\boxed{\overline{m}}$ on top of T .

For $T \in \mathcal{B}$ in the above two cases, we put $\varepsilon_{\overline{m}}(T) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{\overline{m}}^r T \neq \mathbf{0}\}$ and $\varphi_{\overline{m}}(T) = \text{wt}(T) + \varepsilon_{\overline{m}}(T)$. Then we can check that $\mathcal{B} \cup \{\mathbf{0}\}$ is invariant under $\tilde{e}_{\overline{m}}$ and $\tilde{f}_{\overline{m}}$ (we assume $\tilde{e}_i \mathbf{0} = \tilde{f}_i \mathbf{0} = \mathbf{0}$), and hence \mathcal{B} is a \mathfrak{g}_{m+n} -crystal with respect to wt , ε_i , φ_i and \tilde{e}_i, \tilde{f}_i for $i \in I_{m+n}$.

CASE 3. Suppose that $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$ and $\mathcal{B} = \mathbf{T}_{m+n}(a)$ for $0 \leq a \leq m+n$. We regard $\mathbf{T}_{m+n}(a)$ as a subset of $(\mathbf{T}_{m+n}^{\text{sp}})^{\otimes 2}$ by identifying $T \in \mathbf{T}_{m+n}(a)$ with $T^{\text{R}} \otimes T^{\text{L}}$. The signature of $T \in \mathbf{T}_{m+n}(a)$ is invariant under $\tilde{\mathfrak{e}}_i, \tilde{\mathfrak{f}}_i$ for $i \in I_{m+n} \setminus \{\overline{m}\}$ such that $\tilde{\mathfrak{e}}_i T \neq \mathbf{0}$ or $\tilde{\mathfrak{f}}_i T \neq \mathbf{0}$, since $\text{sh}(T^{\text{L}} \rightarrow T^{\text{R}})$ is invariant under $\tilde{\mathfrak{e}}_i, \tilde{\mathfrak{f}}_i$. Next, suppose that $T \in \mathbf{T}_{m+n}(a)$ is given with $\text{sh}(T) = \lambda(a, b, c)$ and $\tilde{\mathfrak{e}}_{\overline{m}} T \neq \mathbf{0}$. If $\tilde{\mathfrak{e}}_{\overline{m}} T = T^{\text{R}} \otimes (\tilde{\mathfrak{e}}_{\overline{m}} T^{\text{L}})$, then $\text{sh}(\tilde{\mathfrak{e}}_{\overline{m}} T) = \lambda(a, b+1, c-1)$ and the signature of $\tilde{\mathfrak{e}}_{\overline{m}} T$ is $(a, b+1)$ since only $c-1$ entries are bumped out from T^{R} when $\tilde{\mathfrak{e}}_{\overline{m}} T^{\text{L}}$ is inserted. Also, if $\tilde{\mathfrak{e}}_{\overline{m}} T = (\tilde{\mathfrak{e}}_{\overline{m}} T^{\text{R}}) \otimes T^{\text{L}}$, then $\text{sh}(\tilde{\mathfrak{e}}_{\overline{m}} T) = \lambda(a, b-1, c)$ and the signature of $\tilde{\mathfrak{e}}_{\overline{m}} T$ is $(a, b-1)$ since the top entry of T^{L} is greater than \overline{m} . Hence $\mathbf{T}_{m+n}(a) \cup \{\mathbf{0}\}$ is invariant under $\tilde{\mathfrak{e}}_{\overline{m}}$. By similar arguments, $\mathbf{T}_{m+n}(a) \cup \{\mathbf{0}\}$ is also invariant under $\tilde{\mathfrak{f}}_{\overline{m}}$. Therefore, $\mathbf{T}_{m+n}(a)$ is a subcrystal of $(\mathbf{T}_{m+n}^{\text{sp}})^{\otimes 2}$ with respect to $\text{wt}, \varepsilon_i, \varphi_i$ and $\tilde{\mathfrak{e}}_i, \tilde{\mathfrak{f}}_i$ for $i \in I_{m+n}$.

Theorem 7.1.

- (1) $\mathbf{T}_{m+n}^{\text{sp}}$ is isomorphic to the crystal of $L_q(\mathfrak{b}_{m+n}, \Lambda_{\overline{m}})$.
- (2) $\mathbf{T}_{m+n}(a)$ is isomorphic to the crystal of $L_q(\mathfrak{g}_{m+n}, \Lambda_{m+n}((1^a), r))$ for $0 \leq a \leq m+n$, where $r = 1$ for $\mathfrak{g} = \mathfrak{c}$ and $r = 2$ otherwise.

Proof. (1) Let $T \in \mathbf{T}_{m+n}^{\text{sp}}$ be given. Let $(\sigma_a)_{a \in \mathbb{J}_{m+n}}$ be sequence of \pm such that $\sigma_a = -$ if and only if a occurs as an entry of T . Then the map sending T to (σ_a) is isomorphism of \mathfrak{b}_{m+n} -crystals from $\mathbf{T}_{m+n}^{\text{sp}}$ to the crystal of the spin representation $L_q(\mathfrak{b}_{m+n}, \Lambda_{\overline{m}})$ (cf. [26, Section 5.4]).

(2) Suppose that $\mathfrak{g} = \mathfrak{c}$. We first claim that $\mathbf{T}_{m+n}(a)$ is connected. We use induction on the number of boxes in $T \in \mathbf{T}_{m+n}(a)$, say $|T|$, to show that T is connected to $H_{(1^a)}$. Suppose that $T \in \mathbf{T}_{m+n}(a)$ is given with $\text{sh}(T) = \mu$. Since $\mathbf{T}_{m+n}(a)$ is a \mathfrak{gl}_{m+n} -crystal, T is connected to a highest weight element in $SST_{\mathbb{J}_{m+n}}(\mu)$ whose columns have \overline{m} as top entries. Hence $\tilde{\mathfrak{e}}_{\overline{m}} T \neq \mathbf{0}$ and $|\tilde{\mathfrak{e}}_{\overline{m}} T| = |T| - 2$, and by induction hypothesis, T is connected to $H_{(1^a)}$.

Next we claim that $\mathbf{T}_{m+n}(a)$ is a regular \mathfrak{c}_{m+n} -crystal, that is, for $J \subset I_{m+n}$ with $|J| \leq 2$ such that $(\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j \in J}$ is of positive definite, it is isomorphic to a crystal of an integrable representation of type $(\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j \in J}$. Since $\mathbf{T}_{m+n}(a)$ is a regular \mathfrak{gl}_{m+n} -crystal, it remains to consider the case when $J = \{\overline{m}, \overline{m-1}\}$. Let $C(T)$ be the connected component of $T \in \mathbf{T}_{m+n}(a)$ with respect to $\tilde{\mathfrak{e}}_i, \tilde{\mathfrak{f}}_i$ for $i = \overline{m}, \overline{m-1}$. It is straightforward to see that $C(T)$ is isomorphic to the crystal of a fundamental representation of type C_2 [26]. Hence $\mathbf{T}_{m+n}(a)$ is a regular \mathfrak{c}_{m+n} -crystal and it is isomorphic to the crystal of an integrable $U_q(\mathfrak{c}_{m+n})$ -module [21]. Since $\mathbf{T}_{m+n}(a)$ is a connected crystal with highest weight $\text{wt}(H_{(1^a)}) = \Lambda_{m+n}((1^a), 1)$, it is isomorphic to the crystal of $L_q(\mathfrak{c}_{m+n}, \Lambda_{m+n}((1^a), 1))$.

Suppose that $\mathfrak{g} = \mathfrak{b}$. By similar arguments as in $\mathfrak{g} = \mathfrak{c}$, we can show that $\mathbf{T}_{m+n}(a)$ is connected for $0 \leq a \leq m+n$ with the highest weight element $H_{(1^a)}$. Since $\mathbf{T}_{m+n}(a)$

is a subcrystal of $(\mathbf{T}_{m+n}^{\text{sp}})^{\otimes 2}$, which is a regular \mathfrak{b}_{m+n} -crystal by (1), $\mathbf{T}_{m+n}(a)$ is also regular with highest weight $\Lambda_{m+n}((1^a), 2)$. Therefore, $\mathbf{T}_{m+n}(a)$ is isomorphic to the crystal of $L_q(\mathfrak{b}_{m+n}, \Lambda_{m+n}((1^a), 2))$.

As an $I_{m|n}$ -colored oriented graph, the crystal of $L_q(\mathfrak{b}_{m+n}^\bullet, \Lambda_{m+n}((1^a), 2))$ is isomorphic to that of $L_q(\mathfrak{b}_{m+n}, \Lambda_{m+n}((1^a), 2))$ (see [1]). This implies that $\mathbf{T}_{m+n}(a)$ is also isomorphic to the crystal of $L_q(\mathfrak{b}_{m+n}^\bullet, \Lambda_{m+n}((1^a), 2))$. \square

Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m+n}$ be given with L as in (6.2). We consider $\mathbf{T}_{m+n}(\lambda, \ell)$ as a subset of

$$(7.1) \quad \begin{cases} \mathbf{T}_{m+n}(\lambda'_1) \otimes \cdots \otimes \mathbf{T}_{m+n}(\lambda'_{L-1}) \otimes \mathbf{T}_{m+n}^{\text{sp}}, & \text{if } \mathfrak{g} = \mathfrak{b} \text{ with } \ell - 2\lambda_1 \text{ odd,} \\ \mathbf{T}_{m+n}(\lambda'_1) \otimes \cdots \otimes \mathbf{T}_{m+n}(\lambda'_L), & \text{otherwise,} \end{cases}$$

by identifying $\mathbf{T} = (T_L, \dots, T_1) \in \mathbf{T}_{m+n}(\lambda, \ell)$ with $T_1 \otimes \cdots \otimes T_L$, and apply $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$ on $\mathbf{T}_{m+n}(\lambda, \ell)$ for $i \in I_{m+n}$.

Lemma 7.2. $\mathbf{T}_{m+n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is invariant under $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$ for $i \in I_{m+n}$.

Proof. (1) Suppose that $\mathfrak{g} = \mathfrak{c}$. Let $\mathbf{T} = T_1 \otimes \cdots \otimes T_\ell \in \mathbf{T}_{m+n}(\lambda, \ell)$ be given. Recall that each T_k can be viewed as $T_k^{\text{R}} \otimes T_k^{\text{L}}$ as an element of a \mathfrak{gl}_{m+n} -crystal. For $1 \leq k \leq \ell$, let $\mathbf{T}^{(k)}$ be given by replacing $T_k^{\text{R}} \otimes T_k^{\text{L}}$ with ${}^{\text{R}}T_k \otimes {}^{\text{L}}T_k$ in \mathbf{T} . Since the column insertion is compatible with the \mathfrak{gl}_{m+n} -crystal structure on $(\mathbb{J}_{m+n})^{\otimes r}$, the map sending $T_k^{\text{R}} \otimes T_k^{\text{L}}$ with ${}^{\text{R}}T_k \otimes {}^{\text{L}}T_k$ commutes with $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$ for $i \in I_{m+n} \setminus \{\overline{m}\}$, and hence so does the map sending \mathbf{T} to $\mathbf{T}^{(k)}$.

Suppose that $\tilde{\mathbf{f}}_i \mathbf{T} \neq \mathbf{0}$ for some $i \in I_{m+n}$ and

$$\tilde{\mathbf{f}}_i \mathbf{T} = T_1 \otimes \cdots \otimes \tilde{\mathbf{f}}_i T_k \otimes \cdots \otimes T_\ell,$$

for some $1 \leq k \leq \ell$. Let $\mathbf{S} = \tilde{\mathbf{f}}_i \mathbf{T} = S_1 \otimes \cdots \otimes S_\ell$. It is clear that $S_j \in \mathbf{T}_{m+n}(\lambda'_j)$ for $1 \leq j \leq \ell$, and $(S_{j+1}, S_j) = (T_{j+1}, T_j)$ is admissible for $1 \leq j \leq \ell - 1$ with $j \neq k - 1, k$.

CASE 1. Suppose that $i \in I_{m+n} \setminus \{\overline{m}\}$. Since $\mathbf{S}^{(k)} = \tilde{\mathbf{f}}_i(\mathbf{T}^{(k)})$, we have

- (i) ${}^{\text{R}}S_k(l) \leq S_{k-1}^{\text{L}}(l)$ for $l \geq 1$, when $k \geq 2$,
- (ii) $S_{k+1}^{\text{R}}(l + \lambda'_k - \lambda'_{k+1}) \leq {}^{\text{L}}S_k(l)$ for $l \geq 1$, when $k \leq \ell - 1$.

Similarly, since $\mathbf{S}^{(j)} = \tilde{\mathbf{f}}_i(\mathbf{T}^{(j)})$ for $j = k - 1, k + 1$, we have

- (i) ${}^{\text{R}}S_{k+1}(l) \leq S_k^{\text{L}}(l)$ for $l \geq 1$, when $k \leq \ell - 1$.
- (ii) $S_k^{\text{R}}(l + \lambda'_{k-1} - \lambda'_k) \leq {}^{\text{L}}S_{k-1}(l)$ for $l \geq 1$, when $k \geq 2$,

This implies that $S_{k+1} \prec S_k$ and $S_k \prec S_{k-1}$. Hence $\tilde{\mathbf{f}}_i \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$.

CASE 2. Suppose that $i = \overline{m}$. First, we claim that $\text{ht}(S_k^{\text{L}}) \leq \text{ht}(T_{k-1}^{\text{L}})$. If $\text{ht}(S_k^{\text{L}}) > \text{ht}(T_{k-1}^{\text{L}})$, then we have $\text{ht}(T_k^{\text{L}}) = \text{ht}(T_{k-1}^{\text{L}})$ with $t_k^{\text{L}}, t_k^{\text{R}} > \overline{m}$ and $t_{k-1}^{\text{L}} = \overline{m}$,

where t_j^L (resp. t_j^R) is the top entry of T_j^L (resp. T_j^R). Since $t_k^L > \overline{m}$, we have ${}^R t_k > \overline{m} = t_{k-1}^L$ by definition of ${}^R T_k$, where ${}^R t_k$ is the top entry of ${}^R T_k$. This contradicts the admissibility of (T_k, T_{k-1}) and hence proves our claim. Now, it is clear that $S_{j+1} \prec S_j$ for $1 \leq j \leq \ell - 1$ since \overline{m} is the smallest one in \mathbb{J}_{m+n} . Hence $\tilde{f}_{\overline{m}} \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$.

By CASE 1 and CASE 2, we have $\tilde{f}_i \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$. The proof of $\tilde{e}_i \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is similar.

(2) Suppose that $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$. Let $\mathbf{T} = T_1 \otimes \cdots \otimes T_L \in \mathbf{T}_{m+n}(\lambda, \ell)$ be given. Suppose that $\tilde{f}_i \mathbf{T} \neq \mathbf{0}$ for some $i \in I_{m+n}$ and

$$\tilde{f}_i \mathbf{T} = T_1 \otimes \cdots \otimes \tilde{f}_i T_k \otimes \cdots \otimes T_L,$$

for some $1 \leq k \leq L$. Let $\mathbf{S} = \tilde{f}_i \mathbf{T} = S_1 \otimes \cdots \otimes S_L$. Then $(S_{j+1}, S_j) = (T_{j+1}, T_j)$ is admissible for $1 \leq j \leq L - 1$ with $j \neq k - 1, k$.

CASE 1. Suppose that $i \in I_{m+n} \setminus \{\overline{m}\}$. By the same argument as in (1), we have $\tilde{f}_i \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$.

CASE 2. Suppose that $i = \overline{m}$. First, assume that $\boxed{\overline{m}}$ has been added on top of T_k^L . In this case, ${}^R S_k = {}^R T_k$ and ${}^L S_k$ is obtained by adding $\boxed{\overline{m}}$ on top of ${}^L T_k$ by construction of ${}^R T_k$ and ${}^L T_k$ (see the proof of Lemma 6.4). Hence $S_{k+1} \prec S_k$ and $S_k \prec S_{k-1}$. Next, assume that $\boxed{\overline{m}}$ has been added on top of T_k^R . In this case, ${}^L S_k = {}^L T_k$ and ${}^R S_k$ is obtained by adding $\boxed{\overline{m}}$ on top of ${}^R T_k$. If $\text{ht}(S_k^R) > \text{ht}(T_{k-1}^L)$, then $\text{ht}(T_k^R) = \text{ht}(T_{k-1}^L)$ and $t_{k-1}^L = \overline{m}$. Since $T_k \prec T_{k-1}$, we have ${}^R t_k = \overline{m}$. But this contradicts the fact that $\boxed{\overline{m}}$ can be added on top of T_k^R . So we have $\text{ht}(S_k^R) \leq \text{ht}(T_{k-1}^L)$. Hence $S_{k+1} \prec S_k$ and $S_k \prec S_{k-1}$.

By CASE 1 and CASE 2, we have $\tilde{f}_i \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$. The proof of $\tilde{e}_i \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is similar. \square

Lemma 7.3. $\mathbf{T}_{m+n}(\lambda, \ell)$ is a connected crystal with highest weight $\Lambda_{m+n}(\lambda, \ell)$.

Proof. (1) Suppose that $\mathfrak{g} = \mathfrak{c}$. Let $\mathbf{H}_{(\lambda, \ell)} = H_{(1^{\lambda'_1})} \otimes \cdots \otimes H_{(1^{\lambda'_\ell})}$. We will prove that any $\mathbf{T} = T_1 \otimes \cdots \otimes T_\ell \in \mathbf{T}_{m+n}(\lambda, \ell)$ is connected to $\mathbf{H}_{(\lambda, \ell)}$ under \tilde{e}_i for $i \in I_{m+n}$ by using induction on $|\mathbf{T}| = \sum_{i=1}^\ell |T_i|$ the sum of the boxes in \mathbf{T} . We may assume that \mathbf{T} is a \mathfrak{gl}_{m+n} -highest weight element.

Choose the smallest $k \geq 1$ such that T_k^R is non-empty. Suppose that there is no such k . Since $T_i^R = \emptyset$, $T_{i+1} \prec T_i$ and $(T_{i+1} \rightarrow (\cdots (T_2 \rightarrow T_1)))$ is a \mathfrak{gl}_{m+n} -highest weight element for $1 \leq i \leq k - 1$, we have $T_i = H_{(1^{\lambda'_i})}$ for $1 \leq i \leq k$, that is, $\mathbf{T} = \mathbf{H}_{(\lambda, \ell)}$. If $k = 1$, then $t_1^L = t_1^R = \overline{m}$. Otherwise, $\tilde{e}_i \mathbf{T} \neq \mathbf{0}$ for some $i \in I_{m+n} \setminus \{\overline{m}\}$. This implies that $\tilde{e}_{\overline{m}} \mathbf{T} \neq \mathbf{0}$. By induction hypothesis, $\tilde{e}_{\overline{m}} \mathbf{T}$ is connected to $\mathbf{H}_{(\lambda, \ell)}$ and so is \mathbf{T} .

Suppose that $k \geq 2$. We have $T_i = H_{(1^{\lambda'_i})}$ for $1 \leq i \leq k-1$. Note that $\text{ht}({}^R T_k) = d$ for some $\lambda'_k < d \leq \lambda'_{k-1}$ and ${}^R T_k(r) \leq T_{k-1}^L(r)$ for $1 \leq r \leq d$. Since $({}^R T_k \rightarrow (T_{k-1} \rightarrow (\cdots \rightarrow T_1)))$ is also a \mathfrak{gl}_{m+n} -highest weight element, it equal to H_μ , where $\mu = (\lambda'_1, \dots, \lambda'_{k-1}, d)'$. In particular, we have ${}^R T_k = H_{(1^d)}$.

We claim that $t_k^R = \overline{m}$. Suppose that $t_k^R > \overline{m}$. Since ${}^R T_k = H_{(1^d)}$, we have $d \geq 2$ and $\overline{m} < {}^L t_k \leq {}^R t'_k$, where ${}^R t'_k$ is the largest entry in ${}^R T_k$. Then $({}^L t_k \rightarrow H_\mu)$ is of shape $\nu = (\lambda'_1, \dots, \lambda'_{k-1}, d, 1)'$ but not a highest weight element H_ν since $\overline{m} < {}^L t_k \leq {}^R t'_k$. This contradicts the fact that \mathbf{T} is a \mathfrak{gl}_{m+n} -highest weight element, and proves our claim, that is, $t_k^L = t_k^R = \overline{m}$. Hence $\tilde{\mathbf{e}}_{\overline{m}} \mathbf{T} \neq \mathbf{0}$ and by induction hypothesis, \mathbf{T} is connected to $\mathbf{H}_{(\lambda, \ell)}$.

(2) Suppose that $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$. Let $\mathbf{H}_{(\lambda, \ell)} = H_{(1^{\lambda'_1})} \otimes \cdots \otimes H_{(1^{\lambda'_L})}$, where we assume that $\lambda'_L = 0$ or the last tensor factor is empty tableau in $\mathbf{T}_{m+n}^{\text{sp}}$ when $\ell - 2\lambda_1$ is odd. Then we can show that $\mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$ is connected to $\mathbf{H}_{(\lambda, \ell)}$ under $\tilde{\mathbf{e}}_i$ for $i \in I_{m+n}$ in almost the same way as in (1). \square

Theorem 7.4. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m+n}$, $\mathbf{T}_{m+n}(\lambda, \ell)$ is isomorphic to the crystal of $L_q(\mathfrak{g}_{m+n}, \Lambda_{m+n}(\lambda, \ell))$.*

Proof. Suppose that $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}$. By Theorem 7.1, $\mathbf{T}_{m+n}(\lambda'_i)$ and $\mathbf{T}_{m+n}^{\text{sp}}$ are regular crystals and so is the crystal (7.1). By Lemma 7.2, $\mathbf{T}_{m+n}(\lambda, \ell)$ is a regular crystal. Hence by Lemma 7.3, it is isomorphic to the crystal of $L_q(\mathfrak{g}_{m+n}, \Lambda_{m+n}(\lambda, \ell))$. Finally, the crystal of $L_q(\mathfrak{b}_{m+n}^\bullet, \Lambda_{m+n}(\lambda, \ell))$ is isomorphic to that of $L_q(\mathfrak{g}_{m+n}, \Lambda_{m+n}(\lambda, \ell))$ as an I_{m+n} -colored oriented graph [1], and hence isomorphic to $\mathbf{T}_{m+n}(\lambda, \ell)$. \square

Let $\mathbb{Z}[P_{m+n}]$ be a group ring of P_{m+n} with a \mathbb{Z} -basis $\{e^\mu \mid \mu \in P_{m+n}\}$, and $\text{ch} L_q(\mathfrak{g}_{m+n}, \Lambda) = \sum_{\mu \in P_{m+n}} \dim L_q(\mathfrak{g}_{m+n}, \Lambda)_\mu e^\mu$ for $\Lambda \in P_{m+n}$. The character of a \mathfrak{g}_{m+n} -crystal is defined in the same way. Put $z = e^{\Lambda_{\overline{m}}}$ and $x_a = e^{\delta_a}$ for $a \in \mathbb{J}_{m+n}$. By Theorem 7.4 we have

Corollary 7.5. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m+n}$, we have*

$$\text{ch} L_q(\mathfrak{g}_{m+n}, \Lambda_{m+n}(\lambda, \ell)) = S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathbb{J}_{m+n}}).$$

Now, we have the following, which is the main result in this section.

Theorem 7.6. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$, we have*

$$\text{ch} L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell)) = S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathbb{J}_{m|n}}).$$

That is, the weight generating function of orthosymplectic tableaux of type \mathfrak{g} and shape (λ, ℓ) is the character of $L_q(\mathfrak{g}_{m|n}, \Lambda_{m+n}(\lambda, \ell))$.

Proof. Considering the classical limit of $L_q(\mathfrak{g}_{m+\infty}, \Lambda_{m+\infty}(\lambda, \ell))$, we have

$$\text{ch}L(\mathfrak{g}_{m+\infty}, \Lambda_{m+\infty}(\lambda, \ell)) = S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathbb{J}_{m+\infty}}) = \sum_{\mu \in \mathcal{P}} K_{\mu(\lambda, \ell)}^{\mathfrak{g}} s_{\mu}(x_{\mathbb{J}_{m+\infty}})$$

by Corollaries 7.5 and 6.13. Hence by Theorem 3.2 (see (3.1)), we have

$$\text{ch}L(\mathfrak{g}_{m|\infty}, \Lambda_{m|\infty}(\lambda, \ell)) = \sum_{\mu \in \mathcal{P}} K_{\mu(\lambda, \ell)}^{\mathfrak{g}} s_{\mu}(x_{\mathbb{J}_{m|\infty}}) = S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathbb{J}_{m|\infty}}).$$

Now $\text{ch}L(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ is obtained by specializing $x_a = 0$ for $a > n + 1$, which is equal to $S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathbb{J}_{m|n}})$. \square

Remark 7.7.

(1) A Weyl-Kac type character formula for $L(\mathfrak{g}_{m|\infty}, \Lambda_{m|\infty}(\lambda, \ell))$ can be obtained by super duality (see also [5] for its detailed expression).

(2) The coefficient $K_{\mu(\lambda, \ell)}^{\mathfrak{g}}$ gives the branching multiplicity with respect to $U_q(\mathfrak{gl}_{m|n})$ -submodules. Even when $n = 0$, our formula for branching multiplicity with respect to $\mathfrak{gl}_m \subset \mathfrak{b}_m, \mathfrak{c}_m$ seems to be new.

(3) We may regard $S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathbb{J}_{m|n}})$ as a natural super-analogue of the irreducible characters over the classical Lie algebras or orthosymplectic analogue of super Schur functions since it is obtained by superizing symmetric functions in $x_{\mathbb{J}_{m+\infty}}$ with respect to $x_{\mathbb{J}_{m|\infty}}$ and then specializing certain variables to 0.

(4) We have more irreducible characters by $S_{(\lambda, \ell)}^{\mathfrak{g}}(x_{\mathcal{A}})$ with other \mathcal{A} 's. For example, when $\mathcal{A} = \tilde{\mathbb{I}}_m^+$ with $m \geq 0$, we have irreducible characters in $\tilde{\mathcal{O}}$ (see (3.2)) corresponding to those in $\mathcal{O}^{int}(m + \infty)$ via T .

7.2. Connection with Kashiwara-Nakashima tableaux. Let us briefly discuss how orthosymplectic tableaux are related with Kashiwara-Nakashima (simply, KN) tableaux [26] when $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}$ and $n = 0$.

Suppose that $\mathfrak{g} = \mathfrak{c}$. Let $T = (T^{\mathbb{L}}, T^{\mathbb{R}}) \in \mathbf{T}_{m+0}(a)$ be given for $0 \leq a \leq m$. Consider $({}^{\mathbb{L}}T, {}^{\mathbb{R}}T)$. Let $\sigma({}^{\mathbb{R}}T)$ be the single-column shaped tableau of height $m - \text{ht}({}^{\mathbb{R}}T)$ with entries $\{1 < \dots < m\}$ such that k appears in $\sigma({}^{\mathbb{R}}T)$ if and only if \bar{k} does not appear in ${}^{\mathbb{R}}T$. Let \tilde{T} be the tableau of height $m - a$ obtained by gluing ${}^{\mathbb{L}}T$ at the bottom of $\sigma({}^{\mathbb{R}}T)$. Then \tilde{T} is a KN tableau of type C_m . For example, when $m = 5$ and $a = 1$, we have

$$T = (T^{\mathbb{L}}, T^{\mathbb{R}}) = \begin{array}{|c|c|} \hline \bar{5} & \bar{4} \\ \hline \bar{3} & \bar{1} \\ \hline \bar{2} & \\ \hline \end{array} \quad ({}^{\mathbb{L}}T, {}^{\mathbb{R}}T) = \begin{array}{|c|c|} \hline \bar{5} & \bar{4} \\ \hline \bar{2} & \bar{3} \\ \hline & \bar{1} \\ \hline \end{array} \quad \tilde{T} = \begin{array}{|c|} \hline 2 \\ \hline 5 \\ \hline \bar{5} \\ \hline \bar{2} \\ \hline \end{array}$$

Now, for $\mathbf{T} = (T_\ell, \dots, T_1) \in \mathbf{T}_{m+0}(\lambda, \ell)$, we have $\tilde{\mathbf{T}} = (\tilde{T}_\ell, \dots, \tilde{T}_1)$, which forms a tableau of shape μ , where $\mu = (\ell - \lambda_\ell, \dots, \ell - \lambda_1)$ and \tilde{T}_k is the k -th column from the right. Then from the admissibility of (T_{k+1}, T_k) for $1 \leq k \leq \ell - 1$, it follows that $\tilde{\mathbf{T}}$ is a KN tableaux of type C_m .

Suppose that $\mathbf{g} = \mathbf{b}$. Let $T = (T^L, T^R) \in \mathbf{T}_{m+0}(a)$ be given for $0 \leq a \leq m$. In this case, we define \tilde{T} as follows: Define $\sigma({}^R T)$ in the same way as in $\mathbf{g} = \mathbf{c}$. We place a single-column shaped tableau of height $a + \text{ht}(T^R) - \text{ht}(T^L)$ with entries 0, at the bottom of $\sigma({}^R T)$ and then glue it with ${}^L T$. Then \tilde{T} is a KN tableau of type B_m with $\text{ht}(\tilde{T}) = m - a$. For example, when $m = 5$ and $a = 1$, we have

$$T = (T^L, T^R) = \begin{array}{|c|c|} \hline & \overline{5} \\ \hline \overline{5} & \overline{4} \\ \hline \overline{3} & \overline{1} \\ \hline \overline{1} & \\ \hline \end{array} \quad ({}^L T, {}^R T) = \begin{array}{|c|c|} \hline & \overline{5} \\ \hline \overline{5} & \overline{4} \\ \hline \overline{1} & \overline{3} \\ \hline & \overline{1} \\ \hline \end{array} \quad \tilde{T} = \begin{array}{|c|} \hline \overline{2} \\ \hline 0 \\ \hline \overline{5} \\ \hline \overline{1} \\ \hline \end{array}$$

If $T \in \mathbf{T}_{m+0}^{\text{sp}}$, then let $\sigma = (\sigma_1, \dots, \sigma_m)$ be a sequence of \pm 's such that $\sigma_k = -$ if and only \overline{k} appears in T . Then σ determines a unique KN tableau of spin shape. Now, we can recover KN tableaux of type B_m from $\mathbf{T}_{m+0}(\lambda, \ell)$ in the same way as in $\mathbf{g} = \mathbf{c}$.

8. CRYSTAL BASE OF A HIGHEST WEIGHT MODULE IN $\mathcal{O}_q^{\text{int}}(m|n)$

We assume that $\mathbf{g} = \mathbf{b}, \mathbf{b}^\bullet, \mathbf{c}$. In this section, we show that $L_q(\mathbf{gl}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ has a unique crystal base for $(\lambda, \ell) \in \mathcal{P}(\mathbf{g})_{m|n}$.

8.1. Crystal structure of \mathcal{V}_q . Let us consider the crystal structure of \mathcal{V}_q in (5.4). We identify $\mathbb{J}_{m|n}$ with the crystal of the natural representation of $U_q(\mathbf{gl}_{m|n})$, where

$$\begin{aligned} \overline{m} &\xrightarrow{\overline{m-1}} \overline{m-1} \xrightarrow{\overline{m-2}} \dots \xrightarrow{\overline{1}} \overline{1} \xrightarrow{0} \frac{1}{2} \xrightarrow{\frac{1}{2}} \frac{3}{2} \xrightarrow{\frac{3}{2}} \dots & (n = \infty), \\ \overline{m} &\xrightarrow{\overline{m-1}} \overline{m-1} \xrightarrow{\overline{m-2}} \dots \xrightarrow{\overline{1}} \overline{1} \xrightarrow{0} \frac{1}{2} \xrightarrow{\frac{1}{2}} \frac{3}{2} \xrightarrow{\frac{3}{2}} \dots \xrightarrow{n-\frac{1}{2}} n - \frac{1}{2} & (n < \infty), \end{aligned}$$

where $\text{wt}(a) = \delta_a$ for $a \in \mathbb{J}_{m|n}$ [2]. The set of finite words with letters in $\mathbb{J}_{m|n}$ is the crystal of the tensor algebra generated by the natural representation of $U_q(\mathbf{gl}_{m|n})$, where we identify each non-empty word $w = w_1 \cdots w_r$ with $w_1 \otimes \cdots \otimes w_r \in (\mathbb{J}_{m|n})^{\otimes r}$.

Remark 8.1. Our convention for a crystal base of a $U_q(\mathbf{gl}_{m|n})$ -module is different from the one in [2], where it is a lower crystal base as a $U_q(\mathbf{gl}_{m|0})$ -module and an upper crystal base as a $U_q(\mathbf{gl}_{0|n})$ -module (cf. Remark 5.1). But we have the same results as in [2]. Hence in our setting, for $w = w_1 \cdots w_r \in (\mathbb{J}_{m|n})^{\otimes r}$, $\tilde{e}_i w$ and $\tilde{f}_i w$ ($i \in I_{0|n}$) are obtained by applying the usual signature rule (see Section 6.3) to w for crystals for symmetrizable Kac-Moody algebras, while for $\tilde{e}_i w$ and $\tilde{f}_i w$ ($i \in I_{m|0} \setminus \{\overline{m}\}$) we

apply the signature rule to the reverse word w^{rev} . Also, for $i = 0$, choose the largest k ($1 \leq k \leq r$) such that $(\beta_0 | \text{wt}(w_k)) \neq 0$. Then $\tilde{e}_0 w$ (resp. $\tilde{f}_0 w$) is obtained by applying \tilde{e}_0 (resp. \tilde{f}_0) to w_k . If there is no such k , then $\tilde{e}_0 w = \mathbf{0}$ (resp. $\tilde{f}_0 w = \mathbf{0}$).

Recall that for a skew Young diagram λ/μ , $SST_{\mathbb{J}_{m|n}}(\lambda/\mu)$ is equipped with an $(I_{m|n} \setminus \{\overline{m}\})$ -colored oriented graph structure ([2, Theorem 4.4]). Here \tilde{e}_i and \tilde{f}_i are defined via the embedding $SST_{\mathbb{J}_{m|n}}(\lambda/\mu) \longrightarrow \bigsqcup_{r \geq 0} (\mathbb{J}_{m|n})^{\otimes r}$, which maps T to $w^{\text{rev}}(T)$ (see Remark 8.1). Then for $\lambda \in \mathcal{P}$, $SST_{\mathbb{J}_{m|n}}(\lambda)$ is isomorphic to the crystal of an irreducible polynomial representation of $U_q(\mathfrak{gl}_{m|n})$ with highest weight $\Lambda_{m|n}(\lambda, 0) \in P_{m|n}$ [2, Theorem 5.1]. We denote by H_λ^\natural the highest weight element with weight $\Lambda_{m|n}(\lambda, 0)$, which is also called a genuine highest weight element [2, Section 4.2].

Let

$$\mathbf{T}_{m|n} = \bigsqcup_{p, q \geq 0} SST_{\mathbb{J}_{m|n}}(1^p) \times SST_{\mathbb{J}_{m|n}}(1^q).$$

By identifying $(T^-, T^+) \in \mathbf{T}_{m|n}$ with $T^- \otimes T^+$ and applying the tensor product rule in Section 5.1, we have an $(I_{m|n} \setminus \{\overline{m}\})$ -colored oriented graph structure on $\mathbf{T}_{m|n}$. Let t^\pm be the top entries in T^\pm . If $t^- = t^+ = \overline{m}$, then we define $\tilde{e}_{\overline{m}}(T^-, T^+) = (S^-, S^+)$, where S^\pm is obtained by removing $\boxed{\overline{m}}$ from T^\pm . Otherwise, we define $\tilde{e}_{\overline{m}}(T^-, T^+) = \mathbf{0}$. Also, if $t^-, t^+ > \overline{m}$, then we define $\tilde{f}_{\overline{m}}(T^-, T^+) = (S^-, S^+)$, where S^\pm is obtained from T^\pm by adding $\boxed{\overline{m}}$ on top of T^\pm . Otherwise, we define $\tilde{f}_{\overline{m}}(T^-, T^+) = \mathbf{0}$. Hence $\mathbf{T}_{m|n}$ has an $I_{m|n}$ -colored oriented graph structure. We also have an $I_{m|n}$ -colored oriented graph structure on $\mathbf{T}_{m|n}^{\text{sp}} := \mathbf{T}_{\mathbb{J}_{m|n}}^{\text{sp}}$, where $\tilde{e}_{\overline{m}}$ and $\tilde{f}_{\overline{m}}$ are defined by adding and removing $\boxed{\overline{m}}$, respectively.

Let $\mathbf{m} = (m_a) \in \mathbf{B}$ be given. Put $d^\pm = \sum_{a \in \pm \mathbb{J}_{m|n}} m_a$. Let $T^\pm(\mathbf{m}) \in SST_{\mathbb{J}_{m|n}}(1^{d^\pm})$ be the unique tableaux such that the entries in $T^\pm(\mathbf{m})$ are a 's with $m_a \neq 0$ counting multiplicity as many as m_a times.

If we regard \mathbf{B} as a crystal of a $U_q(\mathfrak{c}_{m|n})$ -module \mathcal{F}_q , and \mathbf{B}^+ as a crystal of a $U_q(\mathfrak{b}_{m|n})$ -module $\mathcal{F}_{q^2}^+$, then it is straightforward to check the following (see Theorem 5.6 and its proof).

Proposition 8.2. *The maps*

$$\Psi : \mathbf{B} \longrightarrow \mathbf{T}_{m|n}, \quad \Psi^+ : \mathbf{B}^+ \longrightarrow \mathbf{T}_{m|n}^{\text{sp}}$$

given by $\Psi(\mathbf{m}) = (T^-(\mathbf{m}), T^+(\mathbf{m}))$ and $\Psi^+(\mathbf{m}) = T^+(\mathbf{m})$ are bijections which commute with \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$.

From now on, we regard $\mathbf{T}_{m|n}$ (resp. $\mathbf{T}_{m|n}^{\text{sp}}$) as a crystal of \mathcal{F}_q (resp. $\mathcal{F}_{q^2}^+$), where wt , ε_i and φ_i ($i \in I_{m|n}$) are induced from those on \mathbf{B} (resp. \mathbf{B}^+) via Ψ (resp. Ψ^+).

8.2. Highest weight crystal $\mathbf{T}_{m|n}(\lambda, \ell)$. Put

$$\mathbf{T}_{m|n}(a) = \mathbf{T}_{\mathbb{J}_{m|n}}^{\mathfrak{g}}(a), \quad \mathbf{T}_{m|n}(\lambda, \ell) = \mathbf{T}_{\mathbb{J}_{m|n}}^{\mathfrak{g}}(\lambda, \ell),$$

for $a \in \mathbb{Z}_{\geq 0}$ and $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$. We regard

$$\begin{cases} \mathbf{T}_{m|n}(a) \subset \mathbf{T}_{m|n}, & \text{if } \mathfrak{g} = \mathfrak{c}, \\ \mathbf{T}_{m|n}(a) \subset \left(\mathbf{T}_{m|n}^{\text{sp}}\right)^{\otimes 2}, & \text{if } \mathfrak{g} = \mathfrak{b}, \mathfrak{b}^{\bullet}, \end{cases}$$

by identifying T with $T^{\text{L}} \otimes T^{\text{R}}$, and apply \tilde{e}_i and \tilde{f}_i on $\mathbf{T}_{m|n}(a)$ for $i \in I_{m|n}$. Recall that in case of $\mathbf{T}_{m+n}(a)$ in Section 7, we identify T with $T^{\text{R}} \otimes T^{\text{L}}$. This difference is due to the tensor product rule for \tilde{e}_i and \tilde{f}_i ($i \in I_{m|n}$) (see Remark 8.1). But when $n = 0$, $\mathbf{T}_{m+0}(a)$ with respect to \tilde{e}_i and \tilde{f}_i is isomorphic to $\mathbf{T}_{m|0}(a)$ with respect to \tilde{e}_i and \tilde{f}_i for $i \in I_{m+0} = I_{m|0}$.

Lemma 8.3. *Under the above hypothesis, $\mathbf{T}_{m|n}(a) \cup \{\mathbf{0}\}$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$.*

Proof. The proof is almost the same as in the case of $\mathbf{T}_{m+n}(a)$. So we leave the details to the reader. \square

Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$ be given with L as in (6.2). We consider $\mathbf{T}_{m|n}(\lambda, \ell)$ as a subset of

$$(8.1) \quad \begin{cases} \mathbf{T}_{m|n}^{\text{sp}} \otimes \mathbf{T}_{m|n}(\lambda'_{L-1}) \otimes \cdots \otimes \mathbf{T}_{m|n}(\lambda'_1), & \text{if } \mathfrak{g} = \mathfrak{b} \text{ and } \ell - 2\lambda_1 \text{ is odd,} \\ \mathbf{T}_{m|n}(\lambda'_L) \otimes \cdots \otimes \mathbf{T}_{m|n}(\lambda'_1), & \text{otherwise,} \end{cases}$$

by identifying $\mathbf{T} = (T_L, \dots, T_1) \in \mathbf{T}_{m|n}(\lambda, \ell)$ with $T_L \otimes \cdots \otimes T_1$, and apply \tilde{e}_i and \tilde{f}_i on $\mathbf{T}_{m|n}(\lambda, \ell)$ for $i \in I_{m|n}$. We put

$$(8.2) \quad \mathbf{H}_{(\lambda, \ell)}^{\natural} = H_L \otimes \cdots \otimes H_1$$

where $H_k \in SST_{\mathbb{J}_{m|n}}(1^{\lambda'_k})$ ($1 \leq k \leq L$) are unique tableaux such that $(H_L \rightarrow (\cdots (H_2 \rightarrow H_1))) = H_{\lambda}^{\natural}$. We should remark that H_k is not necessarily a highest weight element $H_{(1^{\lambda'_k})}^{\natural}$ in $SST_{\mathbb{J}_{m|n}}(1^{\lambda'_k})$. Here we assume that H_L is the empty tableau in $\mathbf{T}_{m|n}^{\text{sp}}$ if $\mathfrak{g} = \mathfrak{b}$ and $\ell - 2\lambda_1$ is odd.

Theorem 8.4. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$, $\mathbf{T}_{m|n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$. Moreover, $\mathbf{T}_{m|n}(\lambda, \ell)$ is a connected $I_{m|n}$ -colored oriented graph with a highest weight element $\mathbf{H}_{(\lambda, \ell)}^{\natural}$ of weight $\Lambda_{m|n}(\lambda, \ell)$.*

Proof. Let $\mathbf{T} = T_L \otimes \cdots \otimes T_1 \in \mathbf{T}_{m|n}(\lambda, \ell)$ be given. For $1 \leq k \leq L$, we define $\mathbf{T}^{(k)}$ in the same way (see the proof of Lemma 7.2 (1)). Since the column insertion of

$\mathbb{J}_{m|n}$ -semistandard tableaux is compatible with the $\mathfrak{gl}_{m|n}$ -crystal structure (see [22]), we have $\tilde{f}_i(\mathbf{T}^{(k)}) = (\tilde{f}_i \mathbf{T})^{(k)}$. Then by the same argument as in Lemma 7.2, we can show that $\mathbf{T}_{m|n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$.

Next, let us show that $\mathbf{T}_{m|n}(\lambda, \ell)$ is connected. We will prove this only in the case of $\mathfrak{g} = \mathfrak{c}$, since the proof for $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$ is similar. By [2, Theorem 4.8], we may assume that \mathbf{T} is a $\mathfrak{gl}_{m|n}$ -highest weight element, that is, $(T_L \rightarrow (\cdots (T_2 \rightarrow T_1)))$ is a genuine highest weight element. Choose the smallest $k \geq 1$ such that $T_k^{\mathbf{R}}$ is non-empty. If there is no such k , then $\mathbf{T} = \mathbf{H}_{(\lambda, \ell)}^{\natural}$. If $k = 1$, then $t_1^{\mathbf{L}} = t_1^{\mathbf{R}} = \overline{m}$, which implies $\tilde{e}_{\overline{m}} \mathbf{T} \neq \mathbf{0}$. By induction on the number of boxes in \mathbf{T} , $\tilde{e}_{\overline{m}} \mathbf{T}$ is connected to $\mathbf{H}_{(\lambda, \ell)}^{\natural}$ and so is \mathbf{T} .

Suppose that $k \geq 2$. Since $T_i^{\mathbf{R}} = \emptyset$, $T_{i+1} \prec T_i$ for $1 \leq i \leq k-1$ and $(T_\ell \rightarrow (\cdots (T_2 \rightarrow T_1)))$ is a $\mathfrak{gl}_{m|n}$ -highest weight element, we have $T_i(\lambda'_i - r + 1) = \overline{m - r + 1}$ for $1 \leq i \leq k-1$ and $1 \leq r \leq \min\{m, \lambda'_i\}$. Since $\text{ht}({}^{\mathbf{R}}T_k) = d$ for some $\lambda'_k < d \leq \lambda'_{k-1}$ and ${}^{\mathbf{R}}T_k(r) \leq T_{k-1}^{\mathbf{L}}(r)$ for $1 \leq r \leq d$, we also have ${}^{\mathbf{R}}T_k(d - r + 1) = \overline{m - r + 1}$ for $1 \leq r \leq \min\{d, m\}$.

Now, if $t_k^{\mathbf{R}} > \overline{m}$, then we have $d \geq 2$ and $\overline{m} < {}^{\mathbf{L}}t_k \leq {}^{\mathbf{R}}t'_k$, where ${}^{\mathbf{R}}t'_k$ is the largest entry in ${}^{\mathbf{R}}T_k$, and $S = ({}^{\mathbf{L}}t_k \rightarrow ({}^{\mathbf{R}}T_k \rightarrow (T_{k-1} \rightarrow \cdots (T_2 \rightarrow T_1))))$ is of shape $\nu = (\lambda'_1, \dots, \lambda'_{k-1}, d, 1)'$. Since $\overline{m} < {}^{\mathbf{L}}t_k$, the entry in the right-most column of S is not \overline{m} , which implies that $(T_\ell \rightarrow (\cdots (T_2 \rightarrow T_1)))$ is not of the form H_η^{\natural} for some η . This is a contradiction. Therefore, $t_k^{\mathbf{L}} = t_k^{\mathbf{R}} = \overline{m}$, and $\tilde{e}_{\overline{m}} \mathbf{T} \neq \mathbf{0}$. By induction hypothesis, \mathbf{T} is connected to $\mathbf{H}_{(\lambda, \ell)}^{\natural}$. \square

8.3. Crystal base of a highest weight module. First, suppose that $\mathfrak{g} = \mathfrak{c}$, and consider a crystal base $(\mathcal{L}, \mathcal{B})$ of a $U_q(\mathfrak{c}_{m|n})$ -module \mathcal{F}_q in Theorem 5.6. For $a \geq 0$, let $\mathbf{m}(a) \in \mathbf{B}$ be such that $\Psi(\mathbf{m}(a)) = (H_{(1^a)}^{\natural}, \emptyset)$. Then $\mathbf{v}_a := \psi_{\mathbf{m}(a)}|0\rangle$ is a highest weight vector with highest weight $\Lambda_{m|n}((1^a), 1)$. By Theorem 4.2, we have

$$U_q(\mathfrak{c}_{m|n})\mathbf{v}_a \cong L_q(\mathfrak{c}_{m|n}, \Lambda_{m|n}((1^a), 1)).$$

Proposition 8.5. *For $a \geq 0$, let*

$$\begin{aligned} \mathcal{L}(a) &= \sum \mathbb{A} \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_a, \\ \mathcal{B}(a) &= \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_a \pmod{q\mathcal{L}(a)} \} \setminus \{0\}. \end{aligned}$$

where $r \geq 0$, $i_1, \dots, i_r \in I_{m|n}$, and $x = e, f$ for each i_k . Then $(\mathcal{L}(a), \mathcal{B}(a))$ is a crystal base of $L_q(\mathfrak{c}_{m|n}, \Lambda_{m|n}((1^a), 1))$, and the crystal $\mathcal{B}(a)/\{\pm 1\}$ is isomorphic to $\mathbf{T}_{m|n}(a)$.

Proof. Since $\mathbf{v}_a \in \mathcal{L}$, $\mathcal{L}(a) \subset \mathcal{L}$ and it is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$. Also, $\mathcal{B}(a) \subset \mathcal{B}$ and hence it is a pseudo-basis of $\mathcal{L}(a)/q\mathcal{L}(a)$ over \mathbb{Q} since $\mathcal{B}/\{\pm 1\}$

is linearly independent. By Proposition 8.2 and Lemma 8.3, the map

$$(8.3) \quad \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_a \mapsto \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} H_{(1^a)}^{\natural}$$

with $r \geq 0$ and $i_1, \dots, i_r \in I_{m|n}$ is a well-defined weight preserving injection from $\mathcal{B}(a)/\{\pm 1\} \cup \{0\}$ to $\mathbf{T}_{m|n}(a) \cup \{\mathbf{0}\}$, which commutes with \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$. By Theorem 8.4, $\mathbf{T}_{m|n}(a)$ is connected, and hence the map (8.3) is a bijection. Now it follows from Theorem 7.6 that $\text{rank}_{\mathbb{A}} \mathcal{L}(a)_{\mu} = \dim L_q(\mathbf{c}_{m|n}, \Lambda_{m|n}((1^a), 1))_{\mu}$ for all weight μ . This implies that $\mathcal{L}(a)$ is an \mathbb{A} -lattice of $L_q(\mathbf{c}_{m|n}, \Lambda_{m|n}((1^a), 1))$, and hence $(\mathcal{L}(a), \mathcal{B}(a))$ is its crystal base. \square

Next, suppose that $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^{\bullet}$. Consider $\mathcal{F}_{q^2}^+ \otimes \mathcal{F}_{q^2}^+$ as a $U_q(\mathfrak{g}_{m|n})$ -module. For $a \geq 0$, let $\mathbf{m}^+(a) \in \mathbf{B}^+$ be such that $\Psi^+(\mathbf{m}^+(a)) = H_{(1^a)}^{\natural}$.

Lemma 8.6. *For $a \geq 0$, there exists $\mathbf{v}_a \in \mathcal{F}_{q^2}^+ \otimes \mathcal{F}_{q^2}^+$ such that*

- (1) \mathbf{v}_a is a highest weight vector of highest weight $\Lambda_{m|n}((1^a), 2)$,
- (2) $\mathbf{v}_a \in \mathcal{L}^+ \otimes \mathcal{L}^+$ and $\mathbf{v}_a \equiv \psi_{\mathbf{m}^+(a)}|0\rangle \otimes |0\rangle \pmod{q\mathcal{L}^+ \otimes \mathcal{L}^+}$.

Proof. Let $\mathbf{m}^{(1)}, \mathbf{m}^{(2)} \in \mathbf{B}^+$ be given with $\mathbf{m}^{(s)} = (m_{rs})_{r \in \mathbb{J}_{m|n}}$. For convenience, we identify the $(\mathbb{J}_{m|n} \times 2)$ -matrix $[m_{rs}] = [\mathbf{m}^{(1)} : \mathbf{m}^{(2)}]$ with $\psi_{\mathbf{m}^{(1)}}|0\rangle \otimes \psi_{\mathbf{m}^{(2)}}|0\rangle \in \mathcal{F}_{q^2}^+ \otimes \mathcal{F}_{q^2}^+$. Note that $m_{rs} \in \{0, 1\}$ when $|r| = 0$, and $m_{rs} \in \mathbb{Z}_{\geq 0}$ when $|r| = 1$.

Let $a \geq 0$ be given. Put $b = \max\{0, a - m\}$. Let $\mathbf{M}(a)$ be the set of non-negative integral $(\mathbb{J}_{m|n} \times 2)$ -matrices $M = [m_{rs}]$ satisfying the following conditions:

- (1) $m_{rs} = 0$ for $r > \frac{1}{2}$ and $s = 1, 2$,
- (2) $m_{r1} + m_{r2} = 1$ for $\overline{m} \leq r \leq \overline{l+1}$ where $l = \max\{m - a, 0\}$,
- (3) $m_{\frac{1}{2}1} + m_{\frac{1}{2}2} = b$.

Let $M = [m_{rs}] \in \mathbf{M}(a)$ be given. We write $M \overset{\overline{m}}{\rightsquigarrow} M'$ if $[m_{\overline{m}} m_{\overline{m}2}] = [1 \ 0]$ and M' is obtained from M by replacing $[m_{\overline{m}} m_{\overline{m}2}] = [1 \ 0]$ with $[0 \ 1]$. For $i \in \{\overline{m-1}, \dots, \overline{1}\}$, we write $M \overset{i}{\rightsquigarrow} M'$ if $m_{\overline{i+1}1} = 0$, $m_{\overline{i}1} = 1$ and M' is obtained from M by replacing

$$\begin{bmatrix} m_{\overline{i+1}1} & m_{\overline{i}2} \\ m_{\overline{i}1} & m_{\overline{i}2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly, we write $M \overset{0}{\rightsquigarrow} M'$ if $m_{\overline{1}1} = 0$, $m_{\frac{1}{2}1} \geq 1$ and M' is obtained from M by replacing

$$\begin{bmatrix} m_{\overline{1}1} & m_{\overline{1}2} \\ m_{\frac{1}{2}1} & m_{\frac{1}{2}2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ u & v \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} 1 & 0 \\ u-1 & v+1 \end{bmatrix}.$$

Identifying M with $\psi_{\mathbf{m}^{(1)}}|0\rangle \otimes \psi_{\mathbf{m}^{(2)}}|0\rangle$, we have

$$(8.4) \quad e_i M = Q_{M, M'}(q) e_i M',$$

for $M \xrightarrow{i} M'$, where $Q_{M,M'}(q)$ is a monomial in q given by

$$(8.5) \quad Q_{M,M'}(q) = \begin{cases} q, & \text{if } i = \overline{m} \text{ and } \mathfrak{g} = \mathfrak{b}, \\ (-1)^{|\mathbf{m}^{(1)}|+1}q, & \text{if } i = \overline{m} \text{ and } \mathfrak{g} = \mathfrak{b}^\bullet, \\ q^2, & \text{if } i = \overline{m-1}, \dots, \overline{1}, \\ (-1)^{|\mathbf{m}^{(1)}|+1}q^{2\langle\beta_0^\vee, \text{wt}(\mathbf{m}^{(2)})\rangle}, & \text{if } i = 0. \end{cases}$$

Here $|\mathbf{m}^{(s)}|$ denotes the degree of $\mathbf{m}^{(s)}$ or $\psi_{\mathbf{m}^{(s)}}|0\rangle$ (cf. Remark 3.1).

Let $M(a) \in \mathbf{M}(a)$ be such that $m_{r1} = 1$ for $\overline{m} \leq r \leq \overline{l+1}$, and $m_{\frac{1}{2}1} = b$. Then for $M \in \mathbf{M}(a)$, we have

$$(8.6) \quad M(a) = M_0 \xrightarrow{i_1} M_1 \xrightarrow{i_2} \dots \xrightarrow{i_r} M_r = M,$$

for some $r \geq 0$, $i_1, \dots, i_r \in \{\overline{m}, \dots, \overline{1}, 0\}$ and $M_1, \dots, M_{r-1} \in \mathbf{M}(a)$. Put

$$h(M) = r, \quad Q_M(q) = \prod_{k=0}^{r-1} Q_{M_k, M_{k+1}}(q).$$

Note that $M \in \mathbf{M}(a)$ is completely determined by its second column $\mathbf{m}^{(2)}$, and with this identification the $\{\overline{m}, \dots, \overline{1}, 0\}$ -colored graph structure on $\mathbf{M}(a)$ with respect to \xrightarrow{i} coincides with the $\mathfrak{b}_{m|1}$ -crystal structure on $\mathbf{T}_{m|1}^{\text{sp}}$ (see Section 8.1). Then we can check without difficulty that $h(M)$ and $Q_M(q)$ are well defined, that is, independent of a path (8.6) from $M(a)$ to M .

Now, we define

$$\mathbf{v}_a = \sum_{M \in \mathbf{M}(a)} (-1)^{h(M)} Q_M(q) M.$$

Then $\mathbf{v}_a \in \mathcal{L}^+ \otimes \mathcal{L}^+$ and $\mathbf{v}_a \equiv \psi_{\mathbf{m}^+(a)}|0\rangle \otimes |0\rangle \pmod{q\mathcal{L}^+ \otimes \mathcal{L}^+}$. It remains to show that \mathbf{v}_a is a highest weight vector, that is, $e_i \mathbf{v}_a = 0$ for $i \in \{\overline{m}, \dots, \overline{1}, 0\}$.

Consider the pairs (M, M') for $M, M' \in \mathbf{M}(a)$ such that $M \xrightarrow{i} M'$. We see that any $M \in \mathbf{M}(a)$ with $e_i M \neq 0$ belongs to one of these pairs. Since $h(M') = h(M) + 1$ and $Q_{M'}(q) = Q_M(q)Q_{M,M'}(q)$, we have by (8.4)

$$\begin{aligned} e_i \left\{ (-1)^{h(M)} Q_M M + (-1)^{h(M')} Q_{M'} M' \right\} \\ = (-1)^{h(M)} Q_M \{ e_i M - Q_{M,M'}(q) e_i M' \} = 0. \end{aligned}$$

This implies that $e_i \mathbf{v}_a = 0$. □

By Lemma 8.6 and Theorem 4.2, we have

$$U_q(\mathfrak{g}_{m|n}) \mathbf{v}_a \cong L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}((1^a), 2)).$$

Proposition 8.7. *Suppose that $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$. For $a \geq 0$, let*

$$\begin{aligned}\mathcal{L}(a) &= \sum \mathbb{A} \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_a, \\ \mathcal{B}(a) &= \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_a \pmod{q\mathcal{L}(a)} \} \setminus \{0\}.\end{aligned}$$

where $r \geq 0$, $i_1, \dots, i_r \in I_{m|n}$, and $x = e, f$ for each i_k . Then $(\mathcal{L}(a), \mathcal{B}(a))$ is a crystal base of $L_q(\mathfrak{gl}_{m|n}, \Lambda_{m|n}((1^a), 2))$, and the crystal $\mathcal{B}(a)/\{\pm 1\}$ is isomorphic to $\mathbf{T}_{m|n}(a)$ for $a \geq 0$.

Proof. We note that $\mathcal{L}(a) \subset \mathcal{L}^+ \otimes \mathcal{L}^+$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$, and $\mathcal{B}(a) \subset \mathcal{B} \otimes \mathcal{B}$ is a pseudo-basis of $\mathcal{L}^+(a)/q\mathcal{L}^+(a)$ over \mathbb{Q} . Then it follows from the same argument in Proposition 8.5 that $(\mathcal{L}(a), \mathcal{B}(a))$ is a crystal base of $L_q(\mathfrak{gl}_{m|n}, \Lambda_{m|n}((1^a), 2))$. \square

8.4. Main result. Now we are ready to state and prove our main theorem in this paper.

Theorem 8.8. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$, $L_q(\mathfrak{gl}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ has a unique crystal base up to scalar multiplication, and its crystal is isomorphic to $\mathbf{T}_{m|n}(\lambda, \ell)$.*

Proof. Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})_{m|n}$ be given with L as in (6.2). For $1 \leq i \leq L$, let V_i be the $U_q(\mathfrak{gl}_{m|n})$ -submodule of \mathcal{V}_q or $\mathcal{V}_q^{\otimes 2}$ generated by $\mathbf{v}_{\lambda'_i}$ (we assume that $\mathbf{v}_{\lambda'_L} = |0\rangle$ in $\mathcal{F}_{q^2}^+$ if $\mathfrak{g} = \mathfrak{b}$ with $\ell - 2\lambda'_1$ odd). Then V_i is isomorphic to the irreducible $U_q(\mathfrak{gl}_{m|n})$ -module with highest weight $\Lambda_{m|n}((1^{\lambda'_i}), r)$, where $r = 1$ is either 1 or 2. Consider a $U_q(\mathfrak{gl}_{m|n})$ -module $V_{(\lambda, \ell)} = V_L \otimes \cdots \otimes V_1$. Then it is completely reducible and has a crystal base [2]. By Propositions 8.5 and 8.7, we may assume that the crystal lattice of $V_{(\lambda, \ell)}$ is contained in $\mathcal{L}_{(\lambda, \ell)}$, where $\mathcal{L}_{(\lambda, \ell)}$ is $\mathcal{L}^+ \otimes \mathcal{L}(\lambda'_{L-1}) \otimes \cdots \otimes \mathcal{L}(\lambda'_1)$ when $\mathfrak{g} = \mathfrak{b}$ with $\ell - 2\lambda'_1$ odd, and $\mathcal{L}(\lambda'_L) \otimes \cdots \otimes \mathcal{L}(\lambda'_1)$ otherwise.

By the decomposition of $V_{(\lambda, \ell)}$ into irreducible $U_q(\mathfrak{gl}_{m|n})$ -modules (see for example [22, Example 5.8]), there exists a unique $U_q(\mathfrak{gl}_{m|n})$ -highest weight vector $\mathbf{v}_{(\lambda, \ell)}$ in $V_{(\lambda, \ell)}$ (up to scalar multiplication) such that $U_q(\mathfrak{gl}_{m|n})\mathbf{v}_{(\lambda, \ell)}$ is isomorphic to the irreducible $U_q(\mathfrak{gl}_{m|n})$ -module with highest weight $\Lambda_{m|n}(\lambda, \ell)$ and $\mathbf{v}_{(\lambda, \ell)} \not\equiv 0 \pmod{q\mathcal{L}_{(\lambda, \ell)}}$. Since $\mathbf{v}_{(\lambda, \ell)} \in V_{(\lambda, \ell)} = V_L \otimes \cdots \otimes V_1$ and $e_{\overline{m}}V_i = 0$ for $1 \leq i \leq L$, we have $e_{\overline{m}}\mathbf{v}_{(\lambda, \ell)} = 0$. Hence $\mathbf{v}_{(\lambda, \ell)}$ is a $U_q(\mathfrak{gl}_{m|n})$ -highest weight vector and

$$U_q(\mathfrak{gl}_{m|n})\mathbf{v}_{(\lambda, \ell)} \cong L_q(\mathfrak{gl}_{m|n}, \Lambda_{m|n}(\lambda, \ell)),$$

by Theorem 4.2, which also implies $L_q(\mathfrak{gl}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ is a direct summand of $\mathcal{V}_q^{\otimes M}$ for some $M \geq 1$. We also have

$$\mathbf{v}_{(\lambda, \ell)} \equiv \pm \mathbf{H}_{(\lambda, \ell)}^\natural \pmod{q\mathcal{L}_{(\lambda, \ell)}}$$

(see (8.2)). Now, we let

$$\begin{aligned}\mathcal{L}(\lambda, \ell) &= \sum \mathbb{A} \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_{(\lambda, \ell)}, \\ \mathcal{B}(\lambda, \ell) &= \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_{(\lambda, \ell)} \pmod{q\mathcal{L}(\lambda, \ell)} \} \setminus \{0\},\end{aligned}$$

where $r \geq 0$, $i_1, \dots, i_r \in I_{m|n}$, and $x = e, f$ for each i_k .

Since $\mathcal{L}(\lambda, \ell) \subset \mathcal{L}_{(\lambda, \ell)}$, $\mathcal{L}(\lambda, \ell)$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$, and $\mathcal{B}(\lambda, \ell)$ is a pseudo-basis of $\mathcal{L}(\lambda, \ell)/q\mathcal{L}(\lambda, \ell)$ over \mathbb{Q} . By Propositions 8.5 and 8.7, the map

$$(8.7) \quad \Psi_{(\lambda, \ell)} : \mathcal{B}(\lambda, \ell)/\{\pm 1\} \cup \{0\} \longrightarrow \mathbf{T}_{m|n}(\lambda, \ell) \cup \{0\},$$

given by $\tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_{(\lambda, \ell)} \longmapsto \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{H}_{(\lambda, \ell)}^\natural$ with $r \geq 0$ and $i_1, \dots, i_r \in I_{m|n}$ is a well defined injection (see (8.1) and (8.2)), which commutes with \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$.

By Theorem 8.4, $\mathbf{T}_{m|n}(\lambda, \ell)$ is connected, and hence the map (8.7) is a bijection. Finally, by Theorem 7.6, $\mathcal{L}(\lambda, \ell)$ is an \mathbb{A} -lattice of $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$, and hence $(\mathcal{L}(\lambda, \ell), \mathcal{B}(\lambda, \ell))$ is its crystal base.

Finally, the uniqueness of a crystal base of $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ follows from the connectedness of $\mathbf{T}_{m|n}(\lambda, \ell)$ and [2, Lemma 2.7 (iii) and (iv)]. \square

Corollary 8.9. *Each $U_q(\mathfrak{g}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$ has a crystal base.*

Corollary 8.10. *Each highest weight $U_q(\mathfrak{g}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$ is a direct summand of $\mathcal{V}_q^{\otimes M}$ for some $M \geq 1$.*

Remark 8.11.

(1) We expect that an irreducible highest weight $U_q(\mathfrak{d}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$ also has a unique crystal base, but we couldn't prove it in this paper, which is mainly due to the technical difficulty in describing a connected component in the crystal of $\mathcal{V}_q^{\otimes M}$ for $M \geq 2$.

(2) There is another combinatorial character formula for $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ ($\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}$) given in terms of Young bitableaux [28], where there is no crystal theory for superalgebras is used. It would be interesting to find a more explicit connection with $\mathbf{T}_{m|n}(\lambda, \ell)$.

(3) Our presentation of $U_q(\mathfrak{g}_{m|n})$ is with respect to the *standard* Borel subalgebra of $\mathfrak{g}_{m|n}$. There are other Borel subalgebras which are not conjugate under the Weyl group. It is natural to consider a crystal base of $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ with respect to other non-standard Borel subalgebras, where a highest weight changes depending on Borels. We expect that $L_q(\mathfrak{g}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ also has a crystal base with respect to a non-standard Borel, but it gives a different graph structure. This will be discussed in another paper.

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